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FOUNDATIONS, MATHEMATICAL LOGIC

Griss, G. F. C. La mathématique intuitioniste sans négation. Nieuw Arch. Wisk. (3) 3 (1955), 134–142.

A clear and concise explanation of the philosophical ideas which led the author to the construction of his negationless intuitionistic mathematics [cf. MR **8**, 307; **12**, 3; **13**, 310; **14**, 4]. *A. Heyting* (Amsterdam).

★**Prior, A. N. Formal logic.** Oxford, at the Clarendon Press, 1955. ix+329 pp. $5.60.

This book consists of three parts. The first part contains the classical propositional calculus (PC) and the first-order predicate calculus. In Ch. I, PC is treated by means of truth tables. In Ch. II different axiomatizations of PC are discussed, and its completeness is proved by the normal form method. Ch. III contains a discussion of the pure implicational calculus and of Wajsberg's axiom system in terms of C (implication) and O (the false), followed by Quine's completeness proof. There is a section on Lesniewski's protothetic and on the results of Łukasiewicz and Meredith on systems with operational variables. Ch. IV contains sections on the first-order predicate calculus, on protothetic with quantifiers for propositional variables, and on a version of the latter calculus in terms of equivalence. In the second part the author treats the traditional logic from Aristotle to de Morgan, using the modern calculus for clarifying remarks and to compare traditional and modern concepts. Ch. I treats the aristotelian syllogistic in this way; in section 3 there is an interesting discussion of Aristotle's propositions κατὰ πρόσληψιν. Ch. II is on the introduction of negation, conjunction and alternation in syllogistic. The quantification of the predicate is discussed in section 4. Ch. III is on singular and existential propositions; it contains a discussion of Venn's diagrams. A section on the theory of descriptions in Principia Mathematica (PM) is joined quite naturally to this chapter. The third part consists of three chapters on different subjects. Ch. I is on modal logics. Besides Lewis' systems, von Wright's deontic logic is studied in some detail. In Ch. II non-classical logics are discussed. There are sections on three-valued logics and on intuitionist logic. Ch. III is on the logic of classes and relations, following PM, with a section on the paradoxes. The book concludes with a list of more than 60 postulate sets for logical calculi, and a select bibliography. The contents are by no means exhausted by the above review. The author makes many digressions, discussing a great number of questions in ancient as well as modern logic. Formal methods are treated as an instrument more than as the aim of the book, and in the later part formal proofs are often replaced by references to the literature. But anyone interested in some form of logic will find in the book a lot of interesting material. Some critical remarks: (i) Gentzen's work is completely ignored. (ii) The author often discusses the truth of a formula without being aware that this depends upon the interpretation of the symbols, and that it is sometimes questionable whether this interpretation can be defined otherwise than by the axioms which the symbols satisfy. (iii) It was not Brouwer's aim to remove paradoxes, as is said on p. 252. *A. Heyting* (Amsterdam).

★**Hermes, Hans. Vorlesung über Entscheidungsprobleme in Mathematik und Logik.** Aschendorffsche Verlagsbuchhandlung, Münster, 1955. ii+140 pp. DM 10.00.

These lectures give a clear introduction to the following topics: decision methods in 'ordinary' mathematics, especially arithmetic, linear algebra, theory of equations; the theory of primitive recursive (p.r.) functions; Ackermann's interlocked recursive function [Math. Ann. **99** (1928), 118–133] which serves as an example of a computable function which is (i) not p.r., but (ii) recursive, i.e. obtained from p.r. functions by means of substitution and the effective least-number operator; the theory of general recursive functions (defined by systems of equations) which are shown to include recursive functions, though the converse is not proved; Turing machines and Turing computability [Proc. London Math. Soc. (2) **42** (1936), 230–265], which are discussed at length, λ-conversion calculi of Church, and the minimal calculus of Fitch [J. Symb. Logic **9** (1944), 89–94; MR **7**, 45] which are treated more briefly. The (recursive) unsolvability of several problems concerning Turing machines is established by means of the diagonal argument, Kleene's normal form for general recursive functions [Math. Ann. **112** (1936), 727–742] and his enumeration theorem [Trans. Amer. Math. Soc. **53** (1943), 41–73; MR **4**, 126] are given and used in the proof of the undecidability of the predicate calculus.

Reviewer's notes: On p. 12, lines 19–21, the author states that if two real numbers have computable decimal expansions, their sum does not necessarily have a computable decimal expansion. This is false. What the author evidently means is that there is no uniform method of constructing the nth digit of the sum from (a finite number of digits of) the summands, i.e. that there is no computable functional [whose (two) arguments and values range over functions of the integers with values 0, 1, $\cdots$, 9] which maps pairs of computable decimals into (the decimal expansion of) their sum. This is undoubtedly the most natural formulation of the concept 'constructive operation on real numbers' in contrast to the (mathematically also interesting) definition of Specker [J. Symb. Logic **14** (1949), 145–158; MR **11**, 151] to which the author refers. — On pp. 91, 96 etc. the author proves that certain recursively enumerable sets $A(n)$ are not recursive, and states that his proofs are indirect; however, they provide a p.r. function $\pi(n)$ such that $A[\pi(p)] \leftrightarrow R_p[\pi(p)]$, where $R_p(n)$ is a partial recursive enumeration of recursive sets, i.e. $A(n)$ is creative in the sense of Post [Bull. Amer. Math. Soc. **50** (1944), 284–316; MR **6**, 29]; since this is the most one can expect of a proof of non-

recursiveness, it seems best to reserve 'indirect' for proofs which do not yield such a function $\pi(n)$. *G. Kreisel*

Hoo, Shih-hua; and Chen, Chiang-yeh. A note on the 4-valued propositional calculus and the four colour problem. J. Chinese Math. Soc. (N.S.) **1** (1951), 243–246. (Chinese summary)

The authors show how the colorability of a map in four colors may be expressed conveniently in terms of the 4-valued propositional calculus, using the notation of Rosser and Turquette [J. Symb. Logic **10** (1945), 61–82; MR **7**, 185]. This suggests the possibility of applying logical techniques in the solution of the four-color problem. *O. Frink* (University Park, Pa.).

Spector, Clifford. Recursive well-orderings. J. Symb. Logic **20** (1955), 151–163.

This paper is a contribution to the theory of constructive ordinals of Kleene [in particular, Amer. J. Math. **77** (1955), 405–428; MR **17**, 5] and the theory of constructive well-orderings of Markwald [Math. Ann. **127** (1954), 135–149; MR **15**, 771]. Let $n \in O$ mean that the (positive) integer n represents an ordinal in the Church-Kleene notation, $n \in W$ that n is the Gödel number of a recursive well-ordering (in some suitable partial recursive numbering of recursive orderings); for $n \in O$, let $H_n(t)$ be the (complete) predicate associated with n in Kleene's hyper-arithmetic hierarchy [Trans. Amer. Math. Soc. **79** (1955), 312–340; MR **17**, 4]. Further, let $|n|<|m|$ $(|n|=|m|)$ mean that $n \in O$ and $m \in O$ and the ordinal represented by n is less than (equal to) the ordinal represented by m; $a \in O_{|m|}$ means that $|a|<|m|$; $W_{|m|}$ is defined similarly.

The author establishes the fundamental result of the hyper-arithmetic hierarchy that $H_n(t)$, $H_m(t)$ are of the same degree (i.e. recursive in each other) if and only if $|n|=|m|$. Further he obtains the following results: (i) The sets W, O, $\langle\hat{n}, \hat{m}\rangle$ $(|n|=|m|)$, $\langle\hat{n}, \hat{m}\rangle$ $(|n|<|m|)$, $\langle\hat{n}, \hat{m}\rangle$ $(|n|\leqq|m|)$, are all of the same degree; (ii) for each $n \in O$, $H_n(t)$ is of lower degree; (iii) every predicate $P(a)$ which is expressible in both one-function-quantifier forms, i.e.

$$P(a)\leftrightarrow(\alpha)(Ex)R_0[a, \bar{\alpha}(x)]\leftrightarrow(E\alpha)(x)R_1[a, \bar{\alpha}(x)],$$

where α ranges over arbitrary number-theoretical functions, is recursive in some $H_n(t)$; (iv) the sets of (i) are definable by means of one of the one-function-quantifier forms, but not in the dual form. (v) If n^* denotes the successor of n in the Church-Kleene notation, $W_{|a|}$ is recursive in H_{a^*}, $O_{|a|}$ is recursive in $H_{a^{**}}$ though, in general, the converse is false. The position concerning the relations between the degrees of the remaining pairs of sets is this: the author's work yields the relation between $W_{|a|}$ and $O_{|b|}$ but, in the reviewer's opinion, it does not lead to a simple bound $\tau(|a|)$ such that H_a is recursive in $W_{\tau(|a|)}$; Markwald's work (loc. cit.) shows that, for finite $|a|$, H_a is recursive in $W_{\omega|a|}$. The paper concludes with some observations on hyper-degrees.

Reviewer's note: This paper and the papers of Kleene and Markwald cited above, lead to a revaluation of Turing's ordinal logics [Proc. London Math. Soc. (2) **45** (1939), 161–228]. Let $n <_f m$ mean that $f \in W$ and n precedes m in the recursive ordering with number f, and let 'proof by ordinal induction with respect to $<_f$' mean: if $P(0)$ and $(x)[x <_f n \rightarrow P(x)] \rightarrow P(n)$, then $P(n)$ (where 0 is assumed to be the first element in the order $<_f$). Then, on the one hand it is obvious that every true proposition $(x)R(x)$ with recursive R, can be proved by ordinal induction with respect to some $<_f$, $|f|=\omega$: namely, let $P(n)$ be $(x)[x\leqq n\rightarrow R(x)]$, and let $n <_f m$ mean: either (i) $n=0$ and $m\neq 0$ or (ii) $n<m$ and $P(m)$ or (iii) $m<n$ and $\neg P(n)$, with a straightforward extension to arbitrary hyper-arithmetic formulae instead of $(x)R(x)$: thus the proof does not depend on the size of $|f|$, but on the particular notation f. On the other hand, as the author observes, there exists a set K recursive in W, which contains a unique notation for each constructive ordinal, and thus Turing's problem is reopened: to determine, for every true proposition $(x)R(x)$, the least $|f(R)|$, with $f(R) \in K$, such that $(x)R(x)$ can be proved by ordinal induction with respect to $<_{f(R)}$. *G. Kreisel.*

Sodnomov, B. S. Noncontradictoriness of a projective estimate of certain noneffective sets. Uspehi Mat. Nauk (N.S.) **10** (1955), no. 1(63), 155–158. (Russian)

This paper is based on results of P. S. Novikov [Trudy Mat. Inst. Steklov. **38** (1951), 279–316; MR **14**, 234]. The author proves the following theorem and corollary. Theorem: Let $\{S\}$ be a system of sets for which a universal set exists, then the following statement is non-contradictory: Among the sets obtained by applying the axiom of choice (of Gödel's system Σ) to $\{S\}$, there exists a projective set. If K is the universal set and λ the projective set of the theorem, and if K is of class α, then, the author notes, the statement that λ is of class not exceeding $\max(2, \alpha)+2$ is non-contradictory. Corollary: The following statements are non-contradictory: 1. Among the non-Lebesgue-measurable sets obtained by selecting one point from each system of rationalities there exists a set of class not exceeding 3. 2. The well known Hausdorff decomposition of the sphere can be realized by projective sets of class not exceeding 3.

The author uses in the proof of the theorem the single-valued function $f^*(x, t)$, "defined on a certain subset F of a sieve C_0" of unbounded index" constructed by Novikov in the paper cited above. The following lemma is also used. Lemma: The set of lower points of every projective class A_n is of class not higher than A_{n+1}. $M_0(x_1^0, \cdots, x_n^0)$ is called a lower point of a set E in the direction of the axis Ox_i if $M_0 \in E$, but each point $M(x_1^0, \cdots, x_i, \cdots, x_n^0)$ with $x_i<x_i^0$ does not belong to E. *E. J. Cogan.*

Fröhlich, A.; and Shepherdson, J. C. Effective procedures in field theory. Philos. Trans. Roy. Soc. London. Ser. A. **248** (1956), 407–432.

In this paper both the positive results in the subject of the title (Kronecker, etc.) and the pioneer negative result [van der Waerden, Math. Ann. **102** (1930), 738–739] are subordinated to the modern notions of algorithm and computability, which are based on the theory of recursive functions. Going beyond these results, the authors show, among other things, that there exists a field K with an algorithm for the decomposition of polynomials into their irreducible factors (a "splitting algorithm"), and such that no such algorithm exists for a certain simple non-separable extension $K(\alpha)$ of K. Moreover, it turns out that the existence or non-existence of a splitting algorithm does not depend on the field as such but on the way in which it is specified. In other words, the existence of a splitting algorithm does not necessarily carry over from a given field K to every other field K' which is isomorphic to K. However, it does carry over from K to K' if the isomorphism between the two fields is explicit (effectively given) in a sense made precise in the paper. The results described here were summarised previously in Math. Z. **62** (1955), 331–334 [MR **17**, 119]. *A. Robinson.*

Kalicki, Jan; and Scott, Dana. Equational completeness of abstract algebras. Nederl. Akad. Wetensch. Proc. Ser. A. **58**=Indag. Math. **17** (1955), 650–659.

This paper is concerned with general algebras [as defined by G. Birkhoff, Proc. Cambridge Philos. Soc. **31** (1935), 433–454]. A set of identities, such as the commutative and associative laws, is closed, briefly, if it includes all identities which are deducible from it; the set is consistent if it does not include all identities, in particular if it does not include the law $x=y$; and the set is (equationally) complete if it is maximal consistent. For any consistent class of identities, one considers the class of algebras containing more than two elements and satisfying the given identities. A completeness theorem applies. As pointed out by the authors, this is an analogue of the (generalized) Gödel theorem for the lower predicate calculus, and one may add that the concepts and results mentioned so far can in part be subordinated to the corresponding concepts and results of the lower predicate calculus.

The central result of the present paper is the explicit enumeration of the equationally complete classes of algebras which are based on an associative binary operation. It is found that their number is countable. This in turn leads to similar results for abelian groups and lattices, some of which are known from previous investigations. [For corresponding works on general groups and rings, cf. B. H. Neumann, Math. Ann. **114** (1937), 506–525; W. Specht, Math. Z. **52** (1950), 557–589; MR **11**, 711.]
A. Robinson (Toronto, Ont.).

Kalicki, Jan. The number of equationally complete classes of equations. Nederl. Akad. Wetensch. Proc. Ser. A. **58**=Indag. Math. **17** (1955), 660–662.

This is a companion to the paper reviewed above. Whereas it is shown there that the number of equationally complete classes of algebras which are given in terms of an associative binary operation is countable, the author proves in the present paper that if the assumption of associativity is dropped, then the set of equationally complete classes of such algebras possesses the cardinal number of the continuum. In order to prove this result, the author constructs two infinite sequences of identities and shows that there are continuum-many ways of choosing identities from the two sequences so as to obtain consistent classes of identities which are not however compatible with each other. *A. Robinson.*

Farinelli, U.; and Gamba, A. Physics and mathematical logic. Nuovo Cimento (10) **1** (1955), 1152–1158.

The possibility of application of mathematical logic to the investigation of physical problems is discussed; the basic elements of mathematical logic are introduced and examples are given. (Author's summary.) *O. Frink.*

Renaud, Paul. Une échelle de simplicité fondée sur les groupes de symétrie. Rev. Gén. Sci. Pures Appl. **62** (1955), 328–345.

See also: Denjoy p. 591; Kemeny, p. 633; Lehman, p. 634; Samson and Mueller, p. 673.

ALGEBRA

Have

★**Falk, G. Algebra.** Handbuch der Physik. Bd. II., pp. 1–116. Springer-Verlag, Berlin-Göttingen-Heidelberg, 1955. DM 88.00.

The algebra of the title refers to an expository account of linear algebra in Euclidean and Unitary spaces, the representation of finite groups and the rotation group in three-space, and finite-dimensional algebras. There is essentially only one hint of a connection between all this and physics: a class of rings in which there is an abstract Poisson-bracket operation is discussed, and there is later an unconvincing attempt to show how they connect the study of rings with Hamiltonian mechanics.
W. G. Lister (Providence, R.I.).

Schuff, Hans Konrad. Polynome über allgemeinen algebraischen Systemen. Math. Nachr. **13** (1955), 343–366.

K. Dörge has investigated the solutions of equations in indeterminates over abstract algebras [Math. Nachr. **4** (1951), 282–297; MR **12**, 583]. He and H. K. Schuff later extended the results from abstract algebras to similar systems with multivalued partial operators [ibid. **10** (1953), 315–330; MR **15**, 498].

In the paper under review a similar investigation is made with definitions extended still further. In the previous papers the properties which could be demanded of the algebras under discussion were those which could (like the associative and commutative laws) be expressed by identities. In this paper four types of property are precisely defined. Properties expressed by identities are of the first kind. Other kinds include such properties as the existence of inverses or the absence of divisors of zero. Thus fields and similar systems can now be treated. The results are somewhat different from those in the previous papers. *H. A. Thurston* (Cambridge, Mass.).

See also: Kalicki and Scott, p. 571; Kalicki, p. 571.

Combinatorial Analysis

Freund, J. E. Restrictedo ccupancy theory—a generalization of Pascal's triangle. Amer. Math. Monthly **63** (1956), 20–27.

Restricted occupancy theory is concerned with the distribution of r like objects into k like cells with no more than m objects in any cell and with empty cells permitted. Since the condition of any cell with respect to the objects is described by the polynomial $1+x+x^2+\cdots+x^m$, the numbers determined are the coefficients of x^r in

$$(1+x+\cdots+x^m)^k.$$

For $m=1$ they are the binomial coefficients and otherwise are said to be generalizations. The recurrence relation which corresponds to $p_m(x)(p_m(x))^{k-1}=(p_m(x))^k$ with $p_m(x)=1+x+\cdots+x^m$ is proved by mathematical induction (the generating function does not appear) while the simpler one which follows from $(1-x)(p_m(x))^k=(1-x^{m+1})(p_m(x))^{k-1}$ is ignored. Three tables of the numbers appear with $k=0(1)5$ and $m=1$, 2 and 3. *J. Riordan*

Seiden, Esther. Further remark on the maximum number of constraints of an orthogonal array. Ann. Math. Statist. **26** (1955), 759–763.

It is shown that an orthogonal array (81, k, 3, 3) cannot

have more than ten constraints. Use is made of an identity established by R. C. Bose and K. A. Bush [same Ann. **23** (1952), 508–524; MR **14**, 442] and of a lemma proved by the author in a previous paper [ibid. **26** (1955), 132–135; MR **17**, 227]. As to definitions and notation cf. the reviews of these papers. *H. B. Mann.*

Thompson, H. R. Factorial designs with small blocks. New Zealand J. Sci. Tech. Sect. B. 33 (1952), 319–344.

To adapt factorial designs to small block size it is necessary to resort to a high degree of confounding. The authors present a number of such designs together with their analysis. *H. B. Mann* (Columbus, Ohio).

Primrose, E. J. F. Resolvable balanced incomplete block designs. Sankhyā 12 (1952), 137–140.

The author constructs resolvably BIB designs from the quadrics of the PG(3, 2) and PG(3, 3) in the following manner. He first obtains two quadrics $S=0$ and $S'=0$ which have no point in common. The quadrics of the pencil $mS+m'S'=0$ then all have no point in common. The points are the treatments, the quadrics form the blocks, and the blocks of one pencil form a replication. The other replications are obtained by means of collineations. Whether this method would work in any PG(3, s) is not known. *H. B. Mann* (Columbus, Ohio).

Roy, Purnendu Mohon. On the relation between b.i.b. and p.b.i.b. designs. J. Indian Soc. Agric. Statist. **6** (1954), 30–47.

The author develops several methods to construct PBIB from BIB. Examples of these methods are: Deletion of a variety and all blocks containing it from certain BIB. Addition of a BIB and a PBIB. Deletion of one or two sets of blocks each containing a complete replication from certain BIB and deletion of such sets of blocks and all varieties in certain blocks of the set. The inverse processes are also investigated and the limitations of the methods discussed. *H. B. Mann.*

Sprott, D. A. Balanced incomplete block designs and tactical configurations. Ann. Math. Statist. **26** (1955), 752–758.

A balanced incomplete block (BIB) design (v, b, r, k, λ) is an arrangement of v varieties in b blocks of k distinct varieties each, so that each variety is contained in r blocks and every pair of varieties is contained in λ blocks. A complete $\alpha-\beta-k-v$ configuration is an arrangement of v elements in blocks of k so that each set of β elements occurs in exactly α blocks. A Steiner system $S(\beta, k, v)$ is a complete $1-\beta-k-v$ configuration.

The following theorems are proved: (i) The $\alpha-\beta-k-v$ configuration is a BIB design with parameters

$$v,\ b=\alpha\binom{v}{\beta}\Big/\binom{k}{\beta},\ r=\alpha\binom{v-1}{\beta-1}\Big/\binom{k-1}{\beta-1},\ k,\ \lambda=\alpha\binom{v-2}{\beta-2}\Big/\binom{k-2}{\beta-2}.$$

(ii) Every set of u elements occurs in λ_u blocks where $\lambda_u=\alpha\binom{v-u}{\beta-u}\Big/\binom{k-u}{\beta-u}$, $u\leqq\beta$. (iii) If a BIB design $(v, b, r, k, \lambda=\lambda_2)$ has the additional property that every set of u elements occurs exactly λ_u times, then the design splits into two smaller designs with parameters

(1) $v'=v-1,\ b'=b-r,\ r'=r-\lambda_2,\ k'=k,\ \lambda_i'=\lambda_i-\lambda_{i+1}$;

(2) $v''=v-1,\ b''=r,\ r''=\lambda_2,\ k''=k-1,\ \lambda_i''=\lambda_{i+1}$.

(iv) Any $\alpha-\beta-k-v$ configuration with $\beta>2$ splits in the manner described in (iii). (v) The blocks B_i of $(2k, 2r, r, k, \lambda)$ together with the complements of blocks B_i (i.e., blocks B_i' which contain the varieties not contained in blocks B_i), comprise a complete $(3\lambda-r)-3-k-2k$ configuration. (vi) The blocks B_i of $(2k-1, b, 2\lambda, k, \lambda)$ together with the complements of blocks B_i augmented by the element ∞ comprise a complete $(b-3\lambda)-3-k-2k$ configuration. (vii) If $S(2, k, v)$ exists, then a necessary and sufficient condition for it to be a symmetrical design (i.e., to have $b=v$) is $v=k^2-k+1$. (viii) Except for the trivial case $S(3, 3, 4)$, $S(3, k, v)$ is not a symmetrical design. (ix) The design (1) obtained from $S(3, k, v)$ in (iii) is not symmetric unless $k=4$. (x) All designs $S(3, k, nk)$ satisfy the necessary conditions for resolvability. (If the blocks can be grouped so that each group contains one complete replication, the design is called "resolvable". Necessary conditions are that v is divisible by k and $b\geqq v+r-1$.) (xi) The only affine resolvable design (i.e., a resolvable design for which $b=v+r-1$) of the form $S(3, k, nk)$ is $S(3, 4, 8)$. *W. S. Connor.*

Das, M. N. Missing plots and a randomised block design with balanced incompleteness. J. Indian Soc. Agric. Statist. **6** (1954), 58–76.

The paper treats missing plots in a randomized block design. The most general case considered is that of plots missing in any manner except that there is at least one treatment for which no plot is missing. It is suggested that the corresponding normal equations be solved by an iterative method. The procedure is illustrated by a design for eight treatments.

Explicit formulas are presented for the variance of the estimated difference between any two treatment effects for all possible configurations of 3 and 4 missing plots. The special case in which the missing plots form a balanced incomplete block design yields explicit formulas for estimating treatment effects and the variance of differences between them. An illustrative example is given. *W. S. Connor* (Washington, D.C.).

Krishna Iyer, P. V. Some distributions arising in matching problems. J. Indian Soc. Agric. Statist. **6** (1954), 5–29.

The matching problems are as in card matching where two or more decks of cards of arbitrary specification (the number of cards of each color) are compared card by card and "hits" are counted. A hit is given by a prior definition, e.g. the occurrence of the same color at the same time in all packs, in any two, in at least two, etc. The author considers the distribution of hits, first for particular colors, then for any color, for both dealing with replacement (infinite sampling) and without replacement (finite sampling) and obtains (effectively) the known factorial moment generating functions (except that there seems to have been a slip in the symbolic procedure followed to give the variance of the number of hits for two decks). His main interest is in the distribution when each color has a score (a number) and the matching produces a number also, according to various definitions; e.g., for two decks with cards x_r, y_r in the rth place, the matching number is taken as either x_r-y_r, or $|x_r-y_r|$, or $(x_r-y_r)^2$. These are also considered for various modes of dealing the decks, not necessarily the same for each. Various generating functions and moments are determined, and used in standard statistical procedures. *J. Riordan.*

See also: Hillman, p. 575; Ward and Dick, p. 641.

Linear Algebra, Polynomials, Invariants

Have

★ **Mirsky, L. An introduction to linear algebra.** Oxford, at the Clarendon Press, 1955. xi+433 pp. \$5.60.

This textbook in elementary matric theory, although confining its discussion to the real and complex fields, introduces such basic concepts as vector space, linear operator, group, and bilinear form. The treatment does not include the Jordan canonical form nor the spectral theory for finite matrices. However, a number of elementary inequalities involving complex matrices are given and there is a lengthy discussion of power series in a matrix. Otherwise the topics covered by the text may be said to be standard. The presentation is thoroughly elementary and includes a host of exercises.

M. F. Smiley (Iowa City, Ia.).

Flanders, Harley. Methods of proof in linear algebra. Amer. Math. Monthly **63** (1956), 1–15.

The author gives an exposition of various known results which illustrate the "coordinate-free" approach to linear algebra, in which the use of a basis of a vector space is avoided whenever possible, but when necessary, a basis is selected which suits the particular problem under consideration. Most of the paper is devoted to a discussion of how a basis of a finite-dimensional vector space can be selected so that the matrices belonging to a set S of linear transformations (l.t.) all have superdiagonal form. The problem is investigated when S is a commutative set of l.t., a nil associative algebra of l.t., a Lie algebra consisting of nilpotent l.t., or a solvable Lie algebra of l.t. Appropriate restrictions on the base field are introduced in each case. The results in the case of Lie algebras are derived from the theorems of Engel and Lie, and proofs of these theorems are included. *C. W. Curtis.*

Ackerson, R. H. A note on vector spaces. Amer. Math. Monthly **62** (1955), 721–722.

Let V be a finite-dimensional vector space over a field F, and A a linear transformation of V into itself. Then the dimension of the intersection of the range of A with the null space of A is equal to the difference between the dimensions of the range of A and that of A^2 [cf. C. C. MacDuffee, The theory of matrices, Springer, Berlin, 1933, Lemma 8.3, p. 11]. The author gives a number of consequences. *A. F. Ruston* (Sheffield).

Burgerhout, Th. J. On certain linear invariant relations between the elements of a square matrix. Nederl. Akad. Wetensch. Proc. Ser. A. **58**=Indag. Math. **17** (1955), 315–321.

The author is concerned with linear relations among the elements of a square matrix U which hold also for any polynomial $P(U)$. The "grade" of U is the degree of the minimal polynomial of U [this is the "index" of MacDuffee, The theory of matrices, Springer, Berlin, 1933]. The author gives elementary proofs of theorems like the following. (a) The grade of $P(U)$ cannot exceed the grade of U. (b) If the n-by-n matrix U has grade r, then there exists a row vector r_0 with n uniquely defined n-by-n matrices B_l such that

$$V_{\cdot l}=B_l V' r_0' \quad (l=1, \cdots, n).$$

Here $V_{\cdot l}$ is the lth column of V. (c) If the n-by-n matrix is of grade n, and if B is an n-by-n matrix such that $UB=BU'$, then $B=B'$. *G. E. Forsythe.*

Antosiewicz, H. A. A theorem on alternatives for pairs of matrices. Pacific J. Math. **5** (1955), 641–642.

It is shown that the following two versions of the transposition theorem [see A. W. Tucker, Theorems of alternatives for pairs matrices, Symposium on linear inequalities and programming, Washington, D.C., June, 1951, Project Scoop, Rep. no. 10 (1952), pp. 180–181; and Motzkin, Econometrica **19** (1951), 184–185; MR **15**, 857], in which A, B are matrices, A', B' their transposes, x, y, u column vectors and ≥ 0 means ≥ 0 but not $=0$, are easily deducible from each other. 1) Either $A'u>0$, $B'u\geq 0$ for some u or $Ax+By=0$ for some $x\geq 0$, $y\geq 0$. 2) Either $A'u\geq 0$, $B'u\geq 0$ for some u or $Ax+By=0$ for some $x>0$, $y\geq 0$. *T. S. Motzkin* (Los Angeles, Calif.).

Watson, G. N. Some identities associated with a discriminant. Proc. Edinburgh Math. Soc. (2) **10** (1956), 101–107.

Using the facts that the characteristic roots of a real symmetric matrix are real and that the characteristic equation of a symmetric matrix remains invariant under an orthogonal transformation, the author derives the following identity of Kummer:

$$F(a, b, c; u, v, w)\equiv(\alpha\beta\gamma+\sum \alpha u^2)^2+\sum \{2u\beta\gamma-vw(\beta-\gamma) +u(2u^2-v^2-w^2)\}^2+15\sum \{vw\alpha+u(v^2-w^2)\}^2,$$

where

$$\alpha=b-c,\ \beta=c-a,\ \gamma=a-b$$

and

$$27F(a, b, c; u, v, w)\equiv 4(\sum a^2-\sum bc+3\sum u^2)^3 -\{\prod(2a-b-c)+54uvw-9\sum (2a-b-c)u^2\}^2.$$

B. W. Jones (Boulder, Colo.).

Keller, Ott-Heinrich. Eine Bemerkung zur Ausführung der körpertheoretischen Operationen in erträglich vielen Schritten. Math. Z. **63** (1955), 277–285.

The paper is devoted to the practical problem of computing the Galois group of a given equation. A number of suggestions are made which will in some cases reduce the enormous labour normally required for this task. One of the remarks concerns the construction of a primitive element of a field $K(\alpha, \beta)$, where α and β are algebraic over the infinite and separable ground field K. The author illustrates his method, part of which is admittedly only tentative, by determining the Galois group of a particular quintic. *W. Ledermann.*

Dujčev, Jordan. Sur l'irréductibilité des polynômes. Ann. Univ. Sofia Fac. Sci. Phys. Math. Livre 1. **48** (1953/54), 27–32 (1954). (Bulgarian. French summary)

Let $f(z)=a_0z^n+\cdots+a_n$, $\varphi(z)=b_0z^m+\cdots+b_m$ be relatively prime polynomials with integer coefficients. Let $F(z)=f(z)+\varepsilon\lambda\varphi(z)$, where λ is an integer, and $\varepsilon=1$ if $m=n-1$ and $\varepsilon=$ sign a_0b_0 if $m=n-2$. Then there is an M such that $F(z)$ is irreducible in the field of rationals if $|\lambda|>M$ and $m=n-1$, or if $\lambda>M$ and $m=n-2$.

A. W. Goodman (Lexington, Ky.).

Rees, D. A basis theorem for polynomial modules. Proc. Cambridge Philos. Soc. **52** (1956), 12–16.

Let P_n denote the polynomial ring $k[X_1, \cdots, X_n]$ over an infinite field k. The author proves that every finitely generated unitary P_n-module has a k-basis of a certain form. As a corollary he derives anew the generalization of

Hilbert's characteristic function theorem to homogeneous (i.e. graded) finitely generated P_n-modules [Snapper, Amer. J. Math. **69** (1947), 622–652; MR **9**, 173].
E. R. Kolchin (New York, N.Y.).

Jacobsthal, Ernst. Über vertauschbare Polynome. Math. Z. **63** (1955), 243–276.

The composite of two polynomials $f(x)$ and $g(x)$ is defined as $f(g(x))$ and is briefly denoted by fg. This law of composition is associative and, in particular, the iterates $f^0=x$, $f^1=f$, $f^2=ff$, $f^3=fff$, $\cdots$ are unambiguously defined. The polynomials f and g are said to commute if $fg=gf$. Every polynomial commutes with each of its iterates and in some cases with no other polynomial. This suggests the problem of finding the most general solution of $fg=gf$, when g is given. This problem is completely solved for linear and quadratic g and for all binomial polynomials x^r+a.

The author further obtains numerous results related to this question, of which the following are typical: a sequence of commuting polynomials of positive degree is called a V-chain if it contains polynomials of each degree. For example,

$$\text{(I)} \qquad x, x^2, x^3, \cdots$$

is a V-chain. It is proved that no V-chain contains polynomials of the same degree. Before examining the set of all V-chains it is convenient to introduce the concept of similar polynomials: denoting the mapping $x\rightarrow ax+b$ $(a\neq 0)$ by $T(a, b)$ or simply by T the author calls the polynomials f and f_1 similar, if $f_1=T^{-1}fT$, i.e. if $f_1(x)=[f(ax+b)-b]/a$. It is clear that the characteristic properties of a V-chain are invariant under similarity transformations. Accordingly it suffices to consider classes of similar V-chains. The rather surprising result is then that there exist exactly two classes of V-chains which are represented by (I) and by the chain

$$\text{(II)} \qquad x, x^2-2, x^3-3x, \cdots$$

respectively, the latter being similar to the sequence of Chebyshev polynomials. *W. Ledermann* (Manchester).

See also: Piehler, p. 587; Fréchet, p. 603; Roşculeţ, p. 603; Schwarz, p. 614; Madić, p. 666; Lotkin, p. 667; Goodey, p. 667; Heinrich, p. 667.

Lattices

Sikorski, R. On σ-complete Boolean algebras. Bull. Acad. Polon. Sci. Cl. III. **3** (1955), 7–9.

Marczewski raised the question, whether it is possible to characterize the class of all σ-complete Boolean algebras B, which are isomorphic to σ-fields of sets, by means of conditions using at most m variables, m being a fixed cardinal. It is shown that the answer is negative for all cardinals m less than the smallest strongly inaccessible cardinal, provided we restrict ourselves to conditions of the form: for every $x_0, x_1, \cdots, V(x_0, x_1, \cdots)$, where V is such that $V(a_0, a_1, \cdots)$ holds in B if and only if it holds in the smallest relatively complete subalgebra of B which contains $a_0, a_1, \cdots$. *B. Jónsson.*

Takeuchi, Kensuke. The free Boolean σ-algebra with countable generators. J. Math., Tokyo **1** (1953), 77–79.

It is shown that the solution to Birkhoff's Problem 79 is affirmative; i.e., the Borel subsets of the Cantor discontinuum form a free σ-complete Boolean algebra. Another solution, as well as a generalization, was given by L. Rieger [Fund. Math. **38** (1951), 35–52; MR **14**, 347, 1278]. *B. Jónsson* (Berkeley, Calif.).

Tarski, Alfred. A lattice-theoretical fixpoint theorem and its applications. Pacific J. Math. **5** (1955), 285–309.

Theorem 1 states that if f is an increasing function on a complete lattice A to A, then the set of all fixpoints of f is non-empty and is a complete lattice under the inclusion relation of A. More generally, this holds for any commutative set of increasing functions. Now let A be a complete, dense, simply ordered set. Call f quasi-increasing and g quasi-decreasing if $f(\bigcup X)\geqq\bigcap f^*(X)$, $f(\bigcap X)\leqq\bigcup f^*(X)$, $g(\bigcup X)\leqq\bigcup g^*(X)$ and $g(\bigcap X)\geqq\bigcap g^*(X)$. (In case A is a compact real interval, this means that f (g) is upper (lower) semicontinuous on the left and lower (upper) semicontinuous on the right.) If $f(0)\geqq g(0)$ and $f(1)\leqq g(1)$, then the set of all $x\in A$ such that $f(x)=g(x)$ is non-empty and complete. With g constant this generalizes the Weierstrass theorem for continuous real functions. Other applications pertain to complete Boolean algebras and to derivative algebras. By the latter term the author means a Boolean algebra with a unary operation D subject to certain conditions, the archetype being the topological notion of derivative. Thus it is shown that in a complete derivative algebra every element is a disjoint sum of a perfect element and of a scattered element.
B. Jónsson (Berkeley, Calif.).

Davis, Anne C. A characterization of complete lattices. Pacific J. Math. **5** (1955), 311–319.

It is shown that if A is an incomplete lattice, then there exists an increasing function f on A to A which has no fixpoint. This is a converse to Theorem 1 of the paper reviewed above. The question is raised, whether f can always be taken to be join-distributive, and an affirmative answer is given under the additional assumption that every subset of A, whose power is less than the power of A, has a supremum. This applies in particular to countable lattices. It is stated without proof, that the answer is also affirmative for Boolean algebras with ordered bases.
B. Jónsson (Berkeley, Calif.).

Nakamura, Masahiro. Center of closure operators and a decomposition of a lattice. Math. Japon. **3** (1954), 49–52.

This deals with the lattice of all closure operators on a lattice. A product $\phi\psi$ of closure operators may be defined by $x(\phi\psi)=(x\phi)\psi$. This product is itself a closure operator only if $\phi\psi=\phi\cap\psi=\psi\phi$. For this reason the author is led to study the sublattice of those closure operators which commute with all closure operators. This he calls the center of the lattice of closure operators. A lattice whose center contains more than 2 elements is called soluble. The author introduces the concept of the cardinal l-sum of lattices, and proves that a complete lattice L, of dimension greater than 2, is soluble if and only if it may be decomposed into a cardinal l-sum of sublattices. In this case its center is isomorphic to the Boolean algebra with 2^k elements, where k is the cardinal number of the nonsoluble summands in a maximal decomposition of L into a cardinal l-sum. *O. Frink* (University Park, Pa.).

Dubreil, P.; et Croisot, R. Propriétés générales de la résiduation en liaison avec les correspondances de Galois. Collect. Math. **7** (1954), 193–203.

This paper contains an abstract treatment of residuation.

A mapping π of a partly-ordered set X into a partly-ordered set Y and a mapping ϱ of Y into X form a Galois correspondence between X and Y if (i) $x\leqq x'$ implies $\pi x\geqq\pi x'$, (ii) $y\leqq y'$ implies $\varrho y\geqq\varrho y'$, (iii) $x\leqq\varrho\pi x$, and (iv) $y\leqq\pi\varrho y$.

The usual correspondence between fields and groups is of this type. The reviewer notices that the correspondence $\mathfrak{p}\to\mathfrak{p}^{\downarrow}$ and $\mathfrak{q}\to\mathfrak{q}^{\dagger}$ between certain congruences on an algebra and certain congruences on its semi-group of translations [Proc. London Math. Soc. (3) **2** (1952), 175–182; MR **14**, 241] is also of this type.

A mapping π satisfying (i) is residuated re the two relations of partial order if, given any y, there is a maximum x for which $\pi x\geqq y$. If this x is denoted by ϱy, then π and ϱ form a Galois correspondence between X and Y.

The author then generalizes to mappings α of $E\times F$ into G, where E, F, and G are partly-ordered sets, for which

$$e\leqq e' \text{ and } f\leqq f' \text{ implies } \alpha(e,f)\geqq\alpha(e',f').$$

α is left-residuated if the mapping $e\to\alpha(e,f)$ of E into G is residuated for every f of F; similarly for right-residuated. Two mappings analogous to the ϱ above are defined in the obvious way, and a large number of properties which come fairly easily from the definitions are proved.

The ideas are then applied to ordered groupoids and to A-modules [reference is made to R. Croisot, exposé no. 25 of the Séminaire P. Dubreil Fac. Sci. Paris, 1954/1955; MR **17**, 451]. *H. A. Thurston* (Cambridge, Mass.).

Hillman, Abraham P. On the number of realizations of a Hasse diagram by finite sets. Proc. Amer. Math. Soc. **6** (1955), 542–548.

Let Σ be a set of n objects, and let all subsets referred to be subsets of Σ. The 'realizations' referred to in the title consist of families of s subsets. The Hasse diagrams and their realizations may be partially classified, for given n, by s and the total number of inclusions, t. (Evidently $0\leqq t\leqq {}_sC_2$.) If $g(n)$ is a function defined for non-negative integers, the sum $\sum_{i=0}^{n} {}_nC_i g(i)$ is denoted by $Tg(n)$; thus $Tm^n=(m+1)^n$, T is a linear operator, and one can use polynomials in T. The author obtains formulae as functions of n for all Hasse diagrams with $s\leqq 4$ by use of the following theorem. Let D' and D'' be Hasse diagrams. Let D be the diagram formed by placing D' below D'' and drawing segments upward from each maximal vertex of D' to each minimal vertex of D''. Let $A(T)\cdot 1$, $A'(T)\cdot 1$, and $A''(T)\cdot 1$ be the numbers of realizations of D, D', and D'' respectively. Then $A(T)=A'(T)A''(T)$ if either D' has more than one maximal vertex or D'' has more than one minimal vertex; or $A(T)=A'(T)A''(T)\cdot 1$ minus the number of realizations of D with the maximum of D' identified with the minimum of D'', otherwise.

For each s, the sum of all the realization numbers, counting each non-self-dual one twice, yields ${}_{2^n}C_s$, as it should. *S. Gorn* (Philadelphia, Pa.).

Rieger, Ladislav. On Suslin-algebras and their representations. Czechoslovak Math. J. **5(80)** (1955), 99–142. (Russian. English summary)

The author's aim is to give algebraic analogs in Boolean algebras of various notions (A-operation, R-operation etc.) of descriptive set theory in order "to bring certain preparative considerations to a planned new theory of the quantification of predicate variables of mathematical logic" [cf. Rieger, Fund. Math. **38** (1951), 35–52; MR **14**, 347, 1278].

Let B be a Boolean algebra [B.a.] and $k_1k_2\cdots k_n\to b_{k_1k_2\cdots k_n}$ a mapping of all finite sequences of positive integers into B; this is called a S(Suslin)-system $\{b_{k_1\cdots k_n}\}$. One defines $A(b_{k_1\cdots k_n})=\sup_z \bigcap_{n=1}^{\infty} b_{z_1\cdots z_n}$, $z=z_1, \cdots, z_n, \cdots$ running over all infinite sequences of natural numbers. One speaks then of a Suslin algebra (S-algebra or S.a.). If its elements are sets, S.a. is called an S-field. B satisfies the weak [strong] zero-condition provided for each $b_k^\alpha \in B$ ($k<\omega_0$, $\alpha<\omega_1$) such that

$$b_k^\alpha\cap b_k^\beta=0(\alpha<\beta<\omega_1)[\text{resp.}\, b_k^\alpha\supseteq b_k^\beta(\alpha<\beta<\omega_1), \inf_\alpha b_k^\alpha=0],$$

one has $\inf_\alpha \bigcup_k b_k^\alpha=0$. Every free σ-algebra satisfies the weak zero-condition. The dual A-operation is defined in this way: $A^*(b_{k_1\cdots k_n})=(A(a_{k_1}'\cdots a_{k_n}'))'$ (' means the complement).

Let φ be a mapping of a S.a. E into the S.a.B; φ is referred to as S-homomorphism, provided $\varphi A=A\varphi$ and $\varphi(x')=\varphi(x)'$. A δ-ideal I of E is called d-ideal, provided

$$\bigcup_n b_{z_1\cdots z_n} \in I \Rightarrow A^*(b_{z_1\cdots z_n}) \in I$$

for a S-system. Dually, one defines s-ideal. B is weakly distributive [w.d.], provided $\bigcap_{i=k}^{\infty}\bigcup_{k=1}^{\infty} a_{ik}=A(b_{z_1\cdots z_n})$, where $b_{z_1\cdots z_n}=a_{1z_1}\cap a_{2z_2}\cap\cdots\cap a_{nz_n}$. B is strongly distributive [s.d.], provided for any infinite sequence of S-systems $\{a^i{}_{z_1\cdots z_n}\}$ one has $\bigcap_i A(a^i{}_{z_1\cdots z_n})=A(b_{t_1}\cdots b_{t_n})$ with $b_k=a_k{}^1$, $b_{kl}=a_k{}^1\cap a_{kl}{}^1$, $b_{klm}=a_k{}^1\cap a_{kl}{}^1\cap a_m{}^2$, etc. diagonally. The w.d. is not implied by the s.d. There exists a S.a. which is s.d. and satisfies the strong 0-condition; such is the S-field of all subsets of a set. To each cardinal $m>0$ is associated a free s.d. S.a. D_m[resp. a w.d. S.a. $\tilde{D}_m{}^0$ having the strong 0-property] with m free generators; this algebra is unique to within S-isomorphism (Th. 2 [resp. Th. 3]). Every free w.[s.]d.S.a. satisfies the weak 0-condition. Each strong distributive S-algebra B is an S-homomorphic image of an S-field of sets. Without the condition "strong distributive" the statement might not hold even in the case $|A|=\aleph_0$ [cf. Problem 80 in Birkhoff, Lattice theory, Amer. Math. Soc. Colloq. Publ. v. 25, rev. ed. New York, 1948; MR **10**, 673].

Let C_m be the minimal S-field of C-sets of the generalized Cantor set $\{0, 1\}^m$ and which contains all the $F\cap G$-sets. C_m is a free s.d. S.a. with m free generators (T. 4). $C_{\aleph_0}$ is a free w.d.S.a. with s.0-condition (T. 5). *G. Kurepa* (Zagreb).

Gel'fand, M. S. Seminormed lattices and metric spaces. Akad. Nauk Azerbaĭdžan. SSR. Trudy Inst. Fiz. Mat. **4-5** (1952), 174–178. (Russian. Azerbaijanian summary)

The author continues investigations of Wilcox and Smiley [Ann. of Math. **40** (2) (1939), 309–327] and Glivenko [Amer. J. Math. **58** (1936), 799–828] into metrics on lattices. Let S be a lattice and Φ a non-negative, strictly increasing function on S such that $\Phi(a)+\Phi(b)\geqq\Phi(a\vee b)+\Phi(a\wedge b)$. ($\Phi$ is called an upper seminorm on S.) The function $\varrho(a,b)=2\Phi(a\vee b)-\Phi(a)-\Phi(b)$ is a metric on S. For a space X with metric ϱ and $x, y, z \in X$, write xyz if $\varrho(x,z)=\varrho(x,y)+\varrho(y,z)$. Sample theorem: A metric space X becomes a lattice under the definition $a\leqq b$ if and only if uab (u a fixed element of X) and with an upper seminorm that defines the metric if and only if the following conditions are satisfied. 1) For all $a, b, x, c \in X$, the conditions uax, ubx, and acb imply ucx. 2) For all

$a, b \in X$ there exist $s, d \in X$ such that sad, sbd, uds, and asb. 3) For all $x \in X$, uxa and uxb imply uxd (a, b, and d as in 2). Similar results are given for lower seminorms, in which the inequality defining Φ is reversed.

E. Hewitt (Seattle, Wash.).

Watanabe, Sigekatu. On the information theory and metric lattices. I. Rep. Univ. Electro-Commun. **5** (1953), 19–33. (Japanese. English summary)

Let $v(x)$ be a real-valued function defined on a lattice L. Theorem 2 proves the equivalence of the three conditions: 1) $v_0(x, y) = 2v(x \vee y) - v(x) - v(y)$ defines a distance in L; 2) $v(x)$ is monotone increasing and $v(x \vee y) + v(z) \leq v(x) + v(y \vee z)$ for $z \leq x$; 3) $v(x \vee y) + v(x \wedge y) \leq v(x) + v(y)$. Moreover, under these conditions it is shown (theorem 3) that $v(y \vee z) + v(y \wedge z) = v(y) + v(z)$ implies modularity $(x \vee y) \wedge z = x \vee (y \wedge z)$, $x \leq z$. In this way, the author gives some "information" upon the non-modularity of C. E. Shannon's metric lattice metrized by the entropy $v(x) = H(x)$ [The lattice theory of information, Symposium on information theory, London, 1950, pp. 105–107].

K. Yosida.

Watanabe, Sigekatu. Théorie d'information et treillis métrique. II. Rep. Univ. Electro-Commun. **6** (1954), 27–35. (Japanese summary)

The totality of the partitions $\Delta = \{E_i\}_{i=1,2,\cdots}$ of a Boolean algebra A ($I = \bigvee_i E$, $E_i \wedge E_j = 0$ for $i \neq j$) constitutes a lattice A' by the natural semi-order $\Delta' \geqq \Delta$ (Δ' is a refinement of Δ). Let $P(E)$ be a probability measure on A. Then to each partition $\Delta \in A'$, there corresponds a "random variable" x_Δ such that Prob $(x_\Delta \in E_i) = P(E_i) = x_i$. The totality of x_Δ for which $H(x_\Delta) = -\sum_i x_i \log x_i < \infty$ constitutes a lattice L by $x_\Delta \vee x_{\Delta'} = x_{\Delta \vee \Delta'}$, $x_\Delta \wedge x_{\Delta'} = x_{\Delta \wedge \Delta'}$. The semi-order $\Delta' \geqq \Delta$ in L is equivalent to the semi-order $x_{\Delta'} \geqq x_\Delta$ defined by $H(x_\Delta \vee x_{\Delta'}) - H(x_{\Delta'}) = 0$. In this way it is proved that L is a metric lattice with the distance $\varrho(x, y) = 2H(x \vee y) - H(x) - H(y)$.

K. Yosida (Tokyo).

See also: Aubert, p. 583; Lesieur, p. 584; Wolfson, p. 647.

Rings, Fields, Algebras

Martić, Ljubo. Une application de séparateur dans la théorie des nombres entiers algébriques. Hrvatsko Prirod. Društvo. Glasnik Mat.-Fiz. Astr. Ser. II. **10** (1955), 37–40. (Serbo-Croatian summary)

Etant donné un anneau commutatif R, on appelle séparateur de R toute partie S de R dont le complément est multiplicativement stable. Toute réunion d'idéaux premiers est un séparateur. L'auteur énonce par erreur la réciproque de ce résultat, et n'en donne pas de démonstration; cette réciproque est d'ailleurs fausse (prendre pour R l'anneau des entiers et pour S le complément de la suite (4^n)).

P. Samuel (Clermont-Ferrand).

Rees, D. Two classical theorems of ideal theory. Proc. Cambridge Philos. Soc. **52** (1956), 155–157.

Soient A un anneau noethérien, $\mathfrak{a}$ et $\mathfrak{b}$ deux idéaux de A; il existe un entier k tel que, pour tout $n > k$, on ait $\mathfrak{a}^n \cap \mathfrak{b} = (\mathfrak{a}^k \cap \mathfrak{b})\mathfrak{a}^{n-k}$. L'auteur démontre cet intéressant résultat (trouvé indépendemment par E. Artin dans le cas des modules noethériens) en considérant un système de générateurs $(a_1, \cdots, a_m)$ de $\mathfrak{a}$, une indéterminée t, l'anneau noethérien gradué $R = A[a_1 t, \cdots, a_m t, t^{-1}]$, et l'idéal $\mathfrak{b} \cdot A[t, t^{-1}] \cap R$. En particulier, si x n'est pas un diviseur de zero dans A, il existe un entier k tel que $\mathfrak{a}^n : Ax \subset \mathfrak{a}^{n-k}$. Comme autres applications l'auteur donne des démonstrations très simples du théorème sur $\bigcap_{n=1}^{\infty} \mathfrak{a}^n$, et du théorème disant que, dans un anneau d'intégrité noethérien, tout idéal premier isolé d'un idéal principal est minimal.

P. Samuel (Clermont-Ferrand).

Yoshida, Michio. A theorem on Zariski rings. Canad. J. Math. **8** (1956), 3–4.

Soit A un anneau de Zariski, c'est à dire un anneau noethérien muni d'un idéal $\mathfrak{m}$ tel que tout idéal de A soit fermé pour la topologie par les $\mathfrak{m}^n$. Si $\mathfrak{a}$ est un idéal de A, tel que l'idéal $\hat{A}\mathfrak{a}$ engendré par $\mathfrak{a}$ dans le complété $\hat{A}$ de A soit engendré par r éléments, alors $\mathfrak{a}$ est engendré par r éléments. On en déduit que, si $\mathfrak{p}$ est un idéal premier de A et si $\mathfrak{P}$ est un idéal premier isolé de $\hat{A}\mathfrak{p}$, alors on a rang$(\mathfrak{p}) \geq$ rang$(\mathfrak{P})$. Si A est un anneau local, et si la relation dim $(\hat{A}/\mathfrak{P}) +$ dim $(\hat{A}_{\mathfrak{P}}) =$ dim $(\hat{A})$ est vraie pour tout idéal $\mathfrak{P}$ de $\hat{A}$, alors la relation analogue est vraie dans A.

P. Samuel (Clermont-Ferrand).

Motzkin, Theodore S. A proof of Hilbert's Nullstellensatz. Math. Z. **63** (1955), 341–344.

The main result is a proof of the theorem of the title, and a bound on the degrees of the coefficient polynomials therein.

E. R. Kolchin (New York, N.Y.).

Lascu, Alexandru. Sur la division des nombres entiers. Acad. R. P. Romîne. Bul. Şti. Secţ. Şti. Mat. Fiz. **7** (1955), 507–515. (Romanian. Russian and French summaries)

A commutative ring is called Euclidean if a mapping of the ring into the natural numbers exists, leading to a (partial) ordering of the ring elements and to the division algorithm. A partially ordered set is called inductive, provided that every totally ordered subset contains a least element. In the present paper Euclidean and principal ideal rings are characterized not by the existence of any one special mapping, but by a common property of all mappings of the ring into inductive sets. Consider the following two properties of the mapping $\mathfrak{N}$ of an integral domain $\mathfrak{o}$ into an inductive set: (D') For any two elements a, b of $\mathfrak{o}$, $b \neq 0$, there exist elements c and q in $\mathfrak{o}$, such that $\mathfrak{N}(b) \leq \mathfrak{N}(c)$ and either $a = cq$ or $\mathfrak{N}(a - cq) < \mathfrak{N}(b)$. ($D''$) For any elements a and b of $\mathfrak{o}$, $b \neq 0$, there exist elements c, μ, ν such that (i) $\mathfrak{N}(b) \leq \mathfrak{N}(c)$; (ii) if $c = c'd$, $a = a'd$, then ν is prime to c'; (iii) either $a\nu = c\mu$, or $\mathfrak{N}(a\nu - c\mu) < \mathfrak{N}(b)$. The main results can now be formulated as follows. Theorem 1: A necessary and sufficient condition that the integral domain $\mathfrak{o}$ should be Euclidean is that any mapping $\mathfrak{N}$ into an inductive set should have property (D'). Theorem 2: A necessary and sufficient condition for the integral domain $\mathfrak{o}$ to be a principal ideal ring is that any mapping $\mathfrak{N}$ of $\mathfrak{o}$ into an inductive set, should have property (D'').

E. Grosswald (Philadelphia, Pa.).

Popovici, Constantin P. Sur l'unicité de la décomposition en facteurs premiers dans l'anneau des entiers de Gauss. Acad. R. P. Romîne. Bul. Şti. Secţ. Şti. Mat. Fiz. **7** (1955), 517–528. (Romanian. Russian and French summaries)

The uniqueness of the decomposition of Gaussian integers into Gaussian primes is proven without the use of Euclid's algorithm in the ring of Gaussian integers. This property of the Gaussian integers is shown to follow from the uniqueness of the decomposition of rational integers into Gaussian primes. In order to prove the latter, the

author uses induction on the number of rational prime factors of a rational integer and the uniqueness of the factorization of rational primes into Gaussian primes.

E. Grosswald (Philadelphia, Pa.).

Boccioni, Domenico. Semianelli complementarizzabili. Rend. Sem. Mat. Univ. Padova **24** (1955), 474–509.

This paper defines for semi-rings a concept analogous to that of quotient-field. A semi-ring is an algebra S with addition (re which it is a semi-group with cancellation) and multiplication (re which it is a semi-group) and both distributive laws. The necessary and sufficient condition that S can be embedded in a ring A with a 1 such that (i) every element m of a given set M of elements of S for which both multiplicative cancellation laws hold has an inverse in A, and (ii) every element of A is of the form $m^{-1}(a-b)$, where $m \in M$, $a \in S$, and $b \in S$, is that, given any $a \in S$, $b \in S$, and $m \in M$, there exist $a_1 \in S$, $b_1 \in S$, and $m_1 \in M$ such that

$$m_1a+b_1m=m_1b+a_1m.$$

Any two such A's are isomorphic.

This is the main result. In section II the same problem is treated for pseudo-rings, i.e., for semi-rings with the cancellation laws for addition. *H. A. Thurston.*

Leger, George F., Jr. A note on some properties of finite rings. Proc. Amer. Math. Soc. **6** (1955), 968–969.

Let R be a finite ring with the property that the only two-sided ideals of R are: R, R^2, $\cdots$, $R^k=(0)$. The author characterizes these rings as the rings isomorphic with $zI[z]/(f(z), z^k)$, where $I[z]$ denotes the ring of polynomials in z over the ring of integers I and $f(z)=pz-\sum_{i=2}^{k-1} a_iz^i$ $(0\leqq a_i<p)$, for some prime p. This is used to show that $R\cong I/p^kI$ if and only if R is a finite ring such that the ideals of its radical are also ideals in R and the only ideals of R are: R, pR, $\cdots$, $p^kR=(0)$. (Line 16, p. 969, $a_2=0$ should be $a_i=0$.) *S. A. Amitsur.*

Amitsur, S. A. Finite subgroups of division rings. Trans. Amer. Math. Soc. **80** (1955), 361–386.

If a finite group G can be embedded into the multiplicative group of a division ring K, then the submodule $V=V(G)$ of K generated by G is a division ring of dimension f^2 over its center Z which is a finite extension of the prime field. As I. N. Herstein [Pacific J. Math. **1** (1953), 121–126; MR **14**, 1056] pointed out, it follows from a theorem of McLagan-Wedderburn that G must be cyclic if the characteristic of K is positive. However, if K has characteristic 0, then G can be non-abelian, even if G:1 is odd, the minimal order of such a group being 63. There are infinitely many other such groups (Th. 6). — The group G has an absolutely irreducible representation Γ of degree f without fixpoint (i.e., only $\Gamma(1)$ has eigenvalue 1) which is obtained as a component of the regular representation of V over Z in a splitting extension of Z. Linear finite groups without fixpoint have been classified by the reviewer [Abh. Math. Sem. Hamburg. Univ. **11** (1935), 187–220] and by G. Vincent [Comment. Math. Helv. **20** (1947), 117–171; MR **9**, 131]. By a detailed application of classfield theory the groups which are embeddable into division-rings, are sorted out.

Let m, n, r, s, t be natural numbers such that $(m, r)=1$, $s=(r-1, m)$, $t=m/s$, $n=$order of r modulo m. For any prime p dividing m let p^{α_p} be the highest power of p dividing m, n_p be the minimal positive integer satisfying $r^{n_p}\equiv1$ $(mp^{-\alpha_p})$, ν_p the minimal positive integer satisfying $r^{\nu_p}\equiv1$ (p^{α_p}), μ_p the minimal positive integer satisfying $r^{\mu_p}\equiv p^{\mu'}$ $(mp^{-\alpha_p})$ for some integer μ', δ_p the minimal positive integer satisfying $p^{\delta_p}\equiv1$ $(mp^{-\alpha_p})$, and let $\delta_p'=\mu_p\delta_p/n_p$. Theorem 4: The group $G_{m,r}$ generated by the elements A, B with the defining relations $A^m=1$, $B^n=A^t$, $BAB^{-1}=A^r$ and of order mn is embeddable into a division ring if and only if one of the following holds: (1) $(n, t)=(s, t)=1$ and (a) $n=s=2$, $r\equiv-1$ (m) or, (b) for every prime factor q of n there is a prime factor p of m such that $q\nmid n_p$, $p\neq2$, $(q, (p^{\delta_p}-1)/s)=1$, (2) $(n, t)=(s, t)=2$, $n=2n'$, $m=2^\alpha m'$, $s=2s'$, where m', s', n' are odd numbers and $\alpha\geqq2$, $r\equiv-1$ (2^α) and (a) $n=s=2$, $r\equiv-1$ (m) or, (b) for every prime factor q of n there is a primefactor p of m such that $q\nmid n_p$ and either $p\neq2$, $(q, (p^{\delta_p'}-1)/s)=1$ or $p=q=2$, $m/4\equiv\delta_p'\equiv1$ (2). Theorem 7: A finite group G can be embedded into a division ring if and only if it is one of the following types: (1) $G_{m,r}$ as in theorem 4, (2) $SL(2, 3)\times G_{m,r}$, where m and the order of 2 modulo m are odd, (3) the group of order 48 with the generators, A, B and the defining relations $A^2=B^3=(AB)^4$, (4) $SL(2, 5)$. As a side result the finite subgroups of the quaternions are determined anew. *H. Zassenhaus* (Princeton, N.J.).

Herstein, I. N. The Lie ring of a simple associative ring. Duke Math. J. **22** (1955), 471–476.

The author considers a simple ring A and its Lie ring $[A, A]$. He proves that if the characteristic of A is $\neq2$, and U is a Lie ideal of $[A, A]$, then

$$[[[U, U], [U, U]], [[U, U], [U, U]]]=(0),$$

and that if U is such that $[U, U]=0$, then it is contained in the centre of A. He further shows that if the characteristic of A is $\neq2, 3$ and U is a Lie ideal of $[A, A]$ such that $[U, U]$ is in the centre of A then so is U. By applying these results he obtains his main theorem: if A is a simple ring of characteristic $\neq2, 3$, then any proper Lie ideal of $[A, A]$ is contained in the centre of A. (In the meanwhile the case of characteristic 2 and 3 has been settled independently, in as yet unpublished papers, by S. A. Amitsur and by W. F. Baxter.) *J. Levitzki* (Jerusalem).

Nakayama, Tadasi. A remark on finitely generated modules. II. Nagoya Math. J. **9** (1955), 21–23.

[For part I see same J. **3** (1951), 139–140; MR **13**, 313.] Proposition 1 asserts that if I is a non-radical right ideal in a ring R, then R can be embedded in a ring S with 1 in such a way that for some element $a \in I$, $1-a$ is a right zero-divisor in S. Proposition 2 makes a similar assertion about a single element which is not right quasi-regular. There are also some remarks amplifying the preceding note. *I. Kaplansky* (Chicago, Ill.).

Mostowski, A. W.; and Sąsiada, E. On the bases of modules over a principal ideal ring. Bull. Acad. Polon. Sci. Cl. III. **3** (1955), 477–478.

Let R be a principal-ideal ring and G an R-module. If the submodule αG, for some $\alpha \in R$, has a basis, then G has a basis. Corollaries: Since every finitely generated torsion-free R-module has a basis, the same statement holds without the word "torsion-free". If there is an $\alpha \in R$, for which $\alpha G=[0]$, then G has a basis. This is, of course, Prüfer's well-known theorem that an abelian group whose elements have bounded orders is a direct sum of cyclic groups. [See also L. Fuchs, Acta Math. Acad. Sci. Hungar. **3** (1952), 177–195; MR **14**, 945.]

K. A. Hirsch (London).

Leavitt, W. G. Finite dimensional modules. An. Acad. Brasil. Ci. **27** (1955), 241–250.

A ring K with unity is called a dimensional ring if every free module M with a finite number of generators satisfies the two conditions (a) every basis of M has the same length n, and (b) every set of $n+1$ elements of M is dependent over K. This paper gives various conditions that K be a dimensional ring. For example, it is proved that K is dimensional if there exists an integer p such that every set $\{a_i\}$ of p nonzero elements of K satisfies an equation of the form $\sum a_i c_i = 0$ with at least one $a_i c_i \neq 0$.

R. E. Johnson (Northampton, Mass.).

Nikodým, Otton Martin. Sur l'extension des corps algébriques abstraits par le procédé généralisé de Cantor, basé sur les suites générales de Moore-Smith qui contiennent une chaîne confinale. C. R. Acad. Sci. Paris **241** (1955), 1249–1250.

In a previous paper [O. M. Nikodým and St. Nikodým, Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) **17** (1954), 334–339; MR **17**, 232] it was shown how the process of completion of a field by the Cantor method could be generalized to any abstract field F, with the help of a linearly ordered algebraic field V and a linearly ordered set of indices L. It was also shown under what conditions an essentially new field was produced.

The present note extends these results to include any "Moore-Smith partially-ordered set" (a 'directed set') to take the place of the set of indices L. *S. Gorn.*

Fried, Ervin. Fields which can be represented as a quotient field of an integral domain properly contained in them. Eötvös. L. Tud.-Egy. Kiadv. Term.-Tud. Kar Évk. **1952-53**, 27–29 (1954). (Hungarian)

Turán asked the question: characterize the commutative fields which can be represented as the quotient field of an integral domain properly contained in the field. The author proves that the necessary and sufficient condition for a field to have this property is that it should not be an algebraic extension of a finite field. Several applications are discussed. [This paper appears also in a German version, Acta Sci. Math. Szeged **15** (1954), 143–144; MR **16**, 992.] *P. Erdös* (Haifa).

Kasch, Friedrich. Bemerkung zum Hauptsatz der Galoisschen Theorie für Schiefkörper. Arch. Math. **6** (1955), 420–422.

The proof of one half of the fundamental theorem in the Galois theory of skew fields is simplified by adapting an idea of J. Ninot [Arch. Math. **6** (1954), 52–54; MR **16**, 439]. The remainder of the theory can then be developed as in the original paper by H. Cartan [Ann. Sci. Ecole Norm. Sup. (3) **64** (1947), 59–77; MR **9**, 325].

W. Ledermann (Manchester).

Dem'yanov, V. B. On representation of a zero of forms of the form $\sum_{i=1}^{m} a_i x_i^n$. Dokl. Akad. Nauk SSSR (N.S.) **105** (1955), 203–205. (Russian)

Let K be any field. Let A_n denote the order of the multiplicative group of K, modulo nth powers. Let C_n be the smallest integer such that any form $\sum a_i x_i^n$ in more than C_n variables represents 0. Theorem: If -1 is a sum of nth powers in K, then $C_n \leq A_n$. For $n=2$ this was a conjecture of the reviewer [J. Math. Soc. Japan **5** (1953), 200–207; MR **15**, 500], proved by Kneser as reported in the review, and also proved by Tsuzuku [ibid. **6** (1954), 325–331; MR **16**, 669]. Dem'yanov's proof is similar to Kneser's. For the field of p-adic numbers the author notes the value of A_n, observes that C_n may be smaller, and finds the exact value of C_n if $p=2$ and n is a power of 2. An application is given to the solution of congruences $\sum a_i x_i^n \equiv 0$ with integral coefficients. Full proofs are included. *I. Kaplansky* (Chicago, Ill.).

Dem'yanov, V. B. On representation of elements of a complete discrete normed field by forms over this field. Dokl. Akad. Nauk SSSR (N.S.) **105** (1955), 401–404. (Russian)

For characteristic $\neq 2$ it is well-known that a quadratic form which represents 0 also represents every element in the field. The author first notes that this result can be salvaged for characteristic 2 by assuming that the form can be anulled without annulling all the partial derivatives. This leads him to formulate an analogous theorem for forms of higher degree that can be proved for a field K complete in a discrete valuation. One assumes that s given forms in $x_1, \cdots, x_m$ can be annulled so as to make the matrix $\partial f_j / \partial x_i$ have rank s; the conclusion is that the forms can assume any s values $c_1, \cdots, c_s$ in K. For a single form which is a linear combination of nth powers, with n not divisible by the characteristic of K, it is noted that the extra condition is superfluous. *I. Kaplansky.*

Nagahara, Takasi; and Tominaga, Hisao. A note on Galois theory of division rings of infinite degree. Proc. Japan Acad. **31** (1955), 655–658.

Soient D un corps, C son centre, L un sous-corps de D qui est le corps des invariants d'un groupe d'automorphismes $\mathfrak{G}$ de D. Considérons la condition suivante sur $\mathfrak{G}$: les transformés par $\mathfrak{G}$ d'un élément quelconque de D forment un ensemble fini; cette condition intervient par exemple dans un travail récent de Nobusawa [Osaka Math. J. **7** (1955), 1–6; MR **16**, 1084]. Les auteurs montrent que cette condition impose à L des restrictions très fortes: en effet, si L' est le commutant de L dans D, ils prouvent que si C est infini, on a nécessairement $L'=C$, et si C est fini, L' est aussi fini. *J. Dieudonné.*

Deskins, W. E. On the homomorphisms of an algebra onto Frobenius algebras. Pacific J. Math. **5** (1955), 501–511.

Let A be an associative algebra of order n over a field. With a certain choice of basis $e_1, \cdots, e_n$ of A, let a parastrophic matrix Q of A assume a form $\begin{pmatrix} T & 0 \\ 0 & 0 \end{pmatrix}$ with a non-singular matrix T of degree m. Then $e_{m+1}, \cdots, e_n$ are seen to span an ideal B of A which is called a parastrophic ideal of A. The ideal B is called regular if there is an idempotent u in A such that $Au \cup B = A$. It is shown that for a regular parastrophic ideal B the residue algebra $A-B$ is a Frobenius algebra and conversely every Frobenius residue-algebra of A can be obtained in this way. Regular parastrophic ideals of minimal order are studied and in particular a condition is given under which two regular parastrophic ideals of minimal order give always isomorphic (Frobenius) residue-algebras. *T. Nakayama.*

Moriya, Mikao. Theorie der 2-Kohomologiegruppen in diskret bewerteten perfekten Körpern. Math. J. Okayama Univ. **5** (1955), 43–77.

Detailed account of the theory whose main results have already been announced by the author [Proc. Japan. Acad. **30** (1954), 787–790; MR **16**, 1087].

G. P. Hochschild (Berkeley, Calif.).

Auslander, Maurice. On the dimension of modules and algebras. III. Global dimension. Nagoya Math. J. **9** (1955), 67–77.

The terminology is that of Cartan-Eilenberg's "Homological algebra" [Princeton, 1956]. Let R be a ring with 1, and assume that 1 operates as the identity map on all R-modules considered. It is shown first that the (left or right) global dimension of R (i.e., the supremum of the projective dimensions of all R-modules) is the supremum of the projective dimensions of the cyclic R-modules and, if R is not semisimple, exceeds the supremum of the projective dimensions of the ideals of R by exactly 1.

The weak global dimension of R is defined as the largest integer n (or ∞) for which the functor Tor_n^R does not vanish. It is shown that if R is (left or right) Noetherian then its (left or right) global dimension coincides with its weak global dimension. In particular, if R is both left and right Noetherian then its left and right global dimensions are equal.

The remaining results hold under the assumption that R has a nilpotent two-sided ideal N such that R/N is semisimple. The most important conclusions are that, in this case, all of the following coincide: the left projective R-dimension of R/N, the left global dimension of R, the weak global dimension of R, the right global dimension of R, the right projective R-dimension of R/N.

If R_1 and R_2 are algebras over a field K, and each satisfies the above assumption, then the global dimension of $R_1\otimes_K R_2$ is shown to be the sum of the global dimensions of R_1 and R_2, provided that $(R_1/N_1)\otimes_K(R_2/N_2)$ is semisimple. *G. P. Hochschild* (Berkeley, Calif.).

Buchsbaum, D. A. Exact categories and duality. Trans. Amer. Math. Soc. **80** (1955), 1–34.

L'auteur se propose d'étendre le champ d'application du livre "Homological Algebra" de H. Cartan et S. Eilenberg [Princeton Univ. Press, 1956]: au lieu de travailler sur des foncteurs définis sur la catégorie des modules, il se propose de généraliser la théorie à des foncteurs additifs, définis sur des catégories abstraites. Les développements généraux du livre cité deviennent alors applicables à d'autres catégories concrètes (par ex. celle des faisceaux sur un espace topologique) pourvu que certains axiomes soient satisfaits. Le traitement abstrait présente un autre avantage: par l'introduction de la catégorie „duale" d'une catégorie donnée [cf. S. MacLane, Bull. Amer. Math. Soc. **56** (1950), 485–516; MR **14**, 133], il permet de considérer uniquement des foncteurs covariants, car la dualisation donne automatiquement des énoncés et des démonstrations pour les foncteurs contravariants. Le présent mémoire contient des démonstrations détaillées des résultats annoncés par l'auteur dans un Appendice au livre de Cartan-Eilenberg.

Une „catégorie exacte" (exact category), ou „catégorie abélienne" dans la terminologie de Grothendieck (non publié), consiste dans la donnée: 1° d'une collection d'objets, avec un objet privilégié $\varnothing$ (objet nul); 2° pour chaque couple (A, B) d'objets, d'un groupe abélien $H(A, B)$, dont les éléments s'appellent „applications" de A dans B; 3° d'une application bilinéaire

$$\varphi: H(B, C)\times H(A, B)\rightarrow H(A, C),$$

appelée „composition" des applications. Ces données sont astreintes à vérifier 4 axiomes, qui en gros expriment l'associativité de la composition, l'existence d'une application identique $\epsilon H(A, A)$, l'existence d'un „noyau", d'un „conoyau", d'une „image" et d'une „coimage" pour toute application $\epsilon H(A, B)$; de plus $H(\varnothing, \varnothing)=0$. On peut alors parler d'isomorphisme (ou équivalence) et de „suite exacte". A chaque catégorie est associée une catégorie duale, ayant mêmes objets, mais pour laquelle le groupe $H^*(A, B)$ est, par définition, $H(B, A)$, l'application $\varphi^*: H^*(B, C)\times H^*(A, B)\rightarrow H^*(A, C)$ étant définie par

$$\varphi^*(u, v)=\varphi(v, u).$$

Dans la Partie I du travail, on établit une série de conséquences de ces 4 axiomes. La Partie II est consacré à une théorie abstraite, self-duale, de l'homologie. Dans la Partie III, on définit les objets „projectifs" et les objets „injectifs", et on introduit l'axiome V (existence de „sommes directes" finies). Moyennant l'axiome supplémentaire VI (resp. VI*) qui assure que pour tout objet A il existe un épimorphisme $P\rightarrow A$ avec P projectif (resp. que pour tout A il existe un monomorphisme $A\rightarrow Q$ avec Q injectif), la théorie des foncteurs satellites et celle des foncteurs dérivés peut être développée comme dans le livre de Cartan-Eilenberg. La Partie IV signale très brièvement des applications: aux foncteurs $\mathrm{Ext}^n(A, B)$, à l'homologie axiomatique des espaces topologiques, à la dualité de Pontrjagin (celle-ci établit un isomorphisme entre la catégorie des groupes abéliens compacts et la catégorie „duale" de celle des groupes abéliens discrets). *H. Cartan* (Paris).

See also: Fröhlich and Shepherdson, p. 570; Krull, p. 582; Aubert, p. 583; Lesieur, p. 584; Numakura, p. 642; Ballier, p. 643; Dixmier, p. 645; Lazard, p. 645; Wolfson, p. 647; Hoskin, p. 665.

Groups, Generalized Groups

Itô, Noboru. On primitive permutation groups. Acta Sci. Math. Szeged **16** (1955), 207–228.

The main question considered here is the following. Let G be a primitive permutation group, with an abelian normal subgroup N, so that N is necessarily elementary abelian and in its regular representation. If G contains another transitive abelian subgroup A, what can be said about the structure of A?

One answer is that if A is of exponent p^m $(m>1)$ and rank w, then $w\geqq(p^{m-1}-1)/(m-1)$. This generalises a result of Ritt [Trans. Amer. Math. Soc. **24** (1922), 21–30] who proved under the assumption of solubility that A cannot be cyclic unless of order 4. Most of the paper is spent in a detailed discussion of the case when A is of type (p^a, p^b) with $a\neq b$. The only such types actually to occur are (9, 3), (4, 2) and (8, 2); and G is then both doubly transitive and insoluble. There is a considerable overlap with B. Huppert [Arch. Math. **6** (1955), 303–310; MR **16**, 994]. *Graham Higman* (Oxford).

Itô, Noboru. Über die Frattini-Gruppe einer endlichen Gruppe. Proc. Japan Acad. **31** (1955), 327–328.

Let G be a finite group, $F(G)$ its Frattini subgroup, H an accessible (=nachinvariant) subgroup, H' the derived group of H. The author proves the following theorem: H is nilpotent if and only if $F(G)=H'$. The case $H=G$ is well-known, and the case $H=G'$ has been proved recently by Huppert [Math. Z. **60** (1954), 409–434; MR **16**, 332]. *K. A. Hirsch* (London).

Itô, Noboru. Über eine zur Frattini-Gruppe duale Bildung. Nagoya Math. J. 9 (1955), 123–127.

The dual to the Frattini subgroup which the author investigates in this note is the factor group $G/F^*(G)$, where G is a finite group and $F^*(G)$ the group generated by all minimal subgroups of G. In contrast to the Frattini subgroup $F(G)$ this group need not be nilpotent, and it is not even known whether it is necessarily soluble. The author shows that the solubility of $G/F^*(G)$ can be asserted in some special cases. A preliminary result is Theorem 1. Let p be an odd prime and G a finite group whose subgroups of order p are all contained in the centre of G. Then G possesses a normal subgroup N which can be complemented by a p-Sylow subgroup, P, i.e. $G=NP$, $N\cap P=1$. For $p=2$ the theorem ceases to be valid. For if q is an odd prime other than 3, then the special linear group $SL(2, q)$ has only one subgroup of order 2, namely the centre. On the other hand it is well known that $SL(2, q)$ coincides with its derived group.

Now let S be a property of finite groups that is hereditary in subgroups and let G be a group with the property S for which $G/F^*(G)$ is not soluble. Then G contains a subgroup K, with the property S, satisfying the following two conditions: $K/F(K)$ is minimal simple, i.e. the group is simple of composite order but every proper subgroup of it is soluble; $F(K)$ contains $F^*(K)$. The author calls such a subgroup K an S-nuclear group (S-Kerngruppe). Whether or not they exist for a given S, remains an open question. But the following theorem shows that for certain properties S such nuclear groups do not exist. Theorem 2. Let S be the following property: If the prime number p divides the order of G, then (with the possible exception of two prime numbers) the centre of a p-Sylow subgroup P of G shall contain all the elements of order p in G. (If the order of G is even, then 2 shall always be one of the exceptional prime factors.) Then G does not contain S-nuclear subgroups. *K. A. Hirsch* (London).

Suzuki, Michio. On finite groups with cyclic Sylow subgroups for all odd primes. Amer. J. Math. 77 (1955), 657–691.

The object of this paper is to give a description of the structure of those non-solvable finite groups in which all Sylow subgroups of odd order are cyclic and a 2-Sylow subgroup is either (a) a dihedral group, or (b) a generalized quaternion group. The main result can be summarized in the following statement. A group with the above property contains always a normal subgroup $G_1=Z\times L$ of index $\leqq 2$, where Z is a (solvable) group whose Sylow subgroups are all cyclic, and L is isomorphic to a group of type $LF(2, p)$ or $SL(2, p)$, according to the cases (a) or (b). $LF(2, p)$ denotes the group of all linear fractional transformations $x'=(\alpha x+\beta)/(\gamma x+\delta)$ over the prime field F_p of characteristic p, with determinant $\alpha\delta-\beta\gamma=1$, and $SL(2, p)$ is the group all non-singular 2×2 matrices over F_p with determinant 1. The structure of finite groups in which all Sylow subgroups are cyclic has been determined by H. Zassenhaus [Abh. Math. Sem. Hamburg Univ. 11 (1935), 187–220].

As an application of the above theorem the author obtains a complete characterization of those non-solvable finite groups, in which all abelian subgroups are cyclic. *A. Kertész* (Debrecen).

Ledermann, W.; and Neumann, B. H. On the order of the automorphism group of a finite group. I. Proc. Roy. Soc. London. Ser. A. 233 (1956), 494–506.

The authors show the following theorem. There exists a function $f(n)$ such that every group of order $\geqq f(n)$ has at least n automorphisms (in the course of the proof it is shown, for instance, that for $n>4$, $f(n)\leqq (n-1)^{n+[(n-2)\log_2(n-1)]}$, where [ ] denotes "the greatest integer in"; cyclic groups force $f(n)\geqq n$ loglog n). The method of proof is to first dispose of the Abelian case; here it is simple to show that an Abelian group of order $\geqq 2n^2$ has at least n automorphisms just using crude estimates for the Euler ϕ-function. The non-Abelian, and interesting, case is treated by exhibiting the group G as a central extension of $F=G|Z$ by Z, the center of G. The factor set $y_{\varrho,\sigma}\in Z$, $\varrho, \sigma \in F$, generate a subgroup Y of Z. This Y is the all-important object to be studied here. Let m be the order of F. Using the relation $y_{\varrho,\sigma}y_{\varrho\sigma,\tau}=y_{\varrho,\sigma\tau}y_{\sigma,\tau}$ of the factor set, the authors show by induction that Y has at most $(m-1)[\log_2 m]$ generators. If Z has s-generators, the authors construct a new group G^* with center Z^* such that $G^*|Z^*\approx G|Z$, and where 1) Z^* has s generators, 2) Z is of index m^s in Z^*, 3) $z^*\in Z^*$ implies that $(z^*)^m\in Z$, 4) $G^*=X^*Z^*$ such that $Y^*=X^*\cap Z^*$ has exponent dividing m, 5) Y^* is generated by at most $(m-1)[\log_2 m]$ elements. Using a result of Schenkman [Proc. Amer. Math. Soc. 6 (1955), 6–11; MR 16, 671] that every automorphism of Z which leaves Y elementwise fixed can be extended to an outer automorphism of G (in fact this extension is explicitly given by Schenkman) the authors are able, by exhibiting automorphisms of Z^* leaving Y^* elementwise fixed, to obtain outer automorphisms of G^*. These automorphisms admit G; however, they may reduce to the identity on G. The condition that they reduce to the identity is however explicitly found. The argument then proceeds according as Z has many or few generators (the division depending upon whether their number is greater or smaller than $(n-2)[\log_2 (n-1)]$); using the material indicated the authors obtain the result cited at the beginning of this review. *I. N. Herstein.*

Neumann, B. H. Groups with automorphisms that leave only the neutral element fixed. Arch. Math. 7 (1956), 1–5.

Let the group G possess an automorphism α of order 3 which fixes no element other than 0. If G is finite, then every element of G commutes with its α-image; and if every element of G commutes with its α-image, then G is nilpotent of class at most 2. *L. J. Paige.*

Brauer, Richard; and Fowler, K. A. On groups of even order. Ann. of Math. (2) 62 (1955), 565–583.

Let $\mathfrak{G}$ be a finite group of order g. The paper first shows that if $g>2$ is even then $\mathfrak{G}$ has a proper subgroup $\mathfrak{B}$ of order $>g^{\frac{1}{3}}$. If m is the number of involutions in $\mathfrak{G}$ and n denotes g/m, then $\mathfrak{G}$ has a normal subgroup $\neq\mathfrak{G}$ whose index is either 2 or $<[n(n+2)/2]!$. The proof to these and their further refinements, which the paper gives, contains many ingenious, if elementary, computations concerning the order of normalizers, the coefficients of the algebra of conjugate classes, etc. If in particular $\mathfrak{G}$ has no invariant involutions then the $\mathfrak{B}$ can be chosen as the normalizer of a real element, where an element of $\mathfrak{G}$ is called real if it is conjugate to its inverse. It follows also that there exists only a finite number of simple groups in which the normalizer of an involution is isomorphic to a given group.

Let $\mathfrak{G}^*$ be the set of elements $\neq 1$ in $\mathfrak{G}$, and for distinct $X, Y\in\mathfrak{G}^*$ let the distance $d(X, Y)$ be defined to be the smallest number l such that there exist $l+1$ elements $X_0, \cdots, X_l$ in $\mathfrak{G}^*$ with $X_0=X$, $X_l=Y$ and X_{i-1}, X_i

commuting for each $i=1, \cdots, l$. For $X=Y$, set $d(X, Y)=0$. It is shown then that if $\mathfrak{G}$ contains more than one (conjugate) class of involutions, any two involutions in $\mathfrak{G}$ have distance at most 3 and any two elements in $\mathfrak{G}^*$ with even $n(X)$, $n(Y)$ have distance at most 5, where $n(X)$ denotes the order of the normalizer $N(X)$ of X. The numbers 3, 5 here can not be improved. If a real element $G\neq 1$ is transformed to its inverse by an involution J (such J exists whenever $n(G)$ is odd) and if $d(G, J)\geqq 3$ then $\mathfrak{H}=N(G)$ is abelian of odd order and J transforms each element of $\mathfrak{H}$ into its inverse. If here, moreover, $d(G, J)\geqq 4$, then $\mathfrak{H}$ is the normalizer of each of its elements $\neq 1$, $d(H, Z)=\infty$ for $1\neq H\in\mathfrak{H}$, $Z\notin\mathfrak{H}$, the order h of $\mathfrak{H}$ is prime to the index g/h, we have $g=hw(1+Nh)$, $h=1+wt$, where $t\geqq 0$, $w\geqq 1$, $N\geqq 0$ are rational integers, $\mathfrak{H}$ splits and has index w in its normalizer, and, moreover, provided $\mathfrak{H}$ is not normal in $\mathfrak{G}$ we have $h\leqq n(J)+1$ with the involution J as above. The examples of this situation with $h=n(J)+1$ are given by LF(2, 2).

The paper finally deals with the characters of $\mathfrak{G}$. With m as before, $\mathfrak{G}$ has an irreducible real character, different from the 1-character, of degree$\leqq\sqrt{(g(g-1)/(m^2+m))}$, in particular $<n=g/m$. For small n this leads to the existence of a normal subgroup with a factor group of known structure. If G is a real element with $n(G)$ odd, an involution J transforming G into G^{-1}, and if we have $p^v\|n(G)$, $p\nmid n(J)$ for a prime p, then $\mathfrak{G}$ has a p-block of defect $d\leqq v$. If J is an involution transforming no element of order p to its inverse, where p is an odd prime, and if $p^v\|n(J)$, there is again a p-block of defect $d\leqq v$. In both of these, the block can be so chosen as to contain a character of positive defect for 2. The paper closes by giving some relations on the values of characters of $\mathfrak{G}$, useful in constructing the characters, in the case where $\mathfrak{G}$ has a subgroup which is the normalizer of each of its elements $\neq 1$. *T. Nakayama* (Nagoya).

Maurer, I. Contribution à l'étude des groupes à partir du quasi-centre. Com. Acad. R. P. Romîne **5** (1955), 1029–1034. (Romanian. Russian and French summaries)

The quasi-centre $Q(G)$ of a group G is the subgroup generated by all cyclic subgroups of G that are permutable with all the subgroups of G. For groups G whose quasi-centre $Q(G)$ is a maximal abelian subgroup the author proves the following obvious properties: 1) The centre is a proper subgroup of $Q(G)$. 2) If $Q(G)$ is finite, then G is periodic. 3) If G is non-abelian and a direct product, and if the non-abelian components of the decomposition have maximal abelian quasi-centres, then $Q(G)$ is a maximal abelian subgroup of G. *K. A. Hirsch* (London).

Černikov, S. N. On complementation of the Sylow Π-subgroups in some classes of infinite groups. Mat. Sb. N.S. **37(79)** (1955), 557–566. (Russian)

Schur's Factorisation Theorem states that in a finite group G a normal subgroup H whose order is prime to its index is complemented in G, that is, there exists a subgroup K of G such that $G=HK$, $H\cap K=1$. This fact can also be expressed as follows: the normal Sylow Π-subgroups of a finite group are complemented [for the definition of a Sylow Π-subgroup see Kuroš, The theory of groups, 2nd ed., Gostehizdat, Moscow, 1953, § 54; MR **15**, 501; **17**, 124]. The object of the present paper is to extend this result to certain classes of infinite groups. Without any restriction on the group it does not hold; for example, the periodic part of a mixed abelian group need not be a direct summand, and if for such a group Π is taken as consisting of all the prime numbers, then the Sylow Π-subgroup is not complemented.

Whether in all periodic groups the normal Sylow Π-subgroups are complemented or not, is an open problem. Even under the — possibly — stronger condition that the group G is locally finite the author can prove the result only when some further restrictions are imposed on G (see below). But it is one of the main results of the paper that if the words "locally finite" are replaced by "locally normal", then all normal Sylow Π-subgroups are complemented. (Here a group G is called locally normal if the normal subgroup generated by an arbitrary finite subset is finite.) The author also proves that a locally normal group is locally soluble if and only if all its (ordinary) Sylow p-subgroups are complemented in it. This result extends P. Hall's well known theorem on finite soluble groups to arbitrary locally normal groups with soluble finite subgroups.

A more detailed account of the contents follows. In § 1 a new and useful concept is introduced: a Sylow Π-subgroup H of a group G is called arithmetically closed if its index in any subgroup of G containing H, if finite, is not divisible by any prime number in Π. Normal Sylow Π-subgroups are obviously arithmetically closed; so are Sylow Π-subgroups of locally soluble groups and Sylow p-subgroups of arbitrary groups. Theorem 1 gives for locally normal groups a "local" condition: a Sylow Π-subgroup H of a locally normal group G is arithmetically closed in G if and only if $H\cap N$ is an arithmetically closed Sylow Π-subgroup of N, where N is a finite normal subgroup of G. This result is used in § 2 to establish the main tool, namely a "local" theorem for complementation. Theorem 2. An arithmetically closed Sylow Π-subgroup H of a locally normal group G is complemented in G if and only if $H\cap N$ is complemented in N, where N is a finite normal subgroup of G. The proof of this theorem requires the apparatus of completion of projection sets in local systems of subgroups [see Kuroš, loc. cit., § 55]. § 3 then yields the generalisations of Schur's and Hall's theorems. Theorem 3. In a locally normal group G a normal Sylow Π-subgroup is complemented. If the complementary set Π′ (consisting of the prime divisors of the orders of the elements of G that are not contained in Π) is not empty, then every complement of H in G is an arithmetically closed Sylow Π′-subgroup of G. Theorem 4. If a locally normal group G is locally soluble, then every Sylow Π-subgroup of G is complemented in G. Theorem 5. If a locally normal group G has a complemented Sylow p-subgroup for every prime number p, then it is locally soluble. Hence Theorem 6. A locally normal group is locally soluble if and only if all its Sylow subgroups are complemented in it. §§ 4 and 5 deal with more specialised situations. Theorem 7. If a locally finite group G is the product of a Sylow Π-subgroup H with its centraliser $Z(H)$, then H is a direct factor of G. Theorem 8. If in a locally finite group G the product of a normal Sylow Π-subgroup H with its centraliser $Z(H)$ has finite index in G, then H is complemented in G. Theorem 9. If a normal Sylow Π-subgroup H of a locally finite group G is an extension of a complete abelian group with minimal condition for subgroups by a finite group, then H is complemented in G. *K. A. Hirsch* (London).

Gerstenhaber, Murray. On canonical constructions. I. Proc. Nat. Acad. Sci. U.S.A. **41** (1955), 233–236.

The author uses the term canonical to mean unique,

subject to certain conditions. To show the importance of canonical correspondences and canonical constructions, he derives proofs of some classical theorems in these terms, and involving also the notion of inner automorphism, which he generalizes from group theory. If S' is a structure derived from a structure S, then an automorphism of S' derived from one of S is called "inner". He first obtains the following four results.

1. Let S' be a structure derived from the structure S, and let there be a canonical isomorphism of S onto a substructure Q of S'. Let T be a structure isomorphic to S, and let T' be the structure derived from T in the same manner as S' from S. Then there exists a canonical isomorphism of T onto a substructure R of T' such that, if ϕ is any isomorphism of S onto T and ϕ' the isomorphism it induces of (S, S') onto (T, T'), then $\phi'(Q)=R$, and the canonical isomorphism of T onto R is the image under ϕ' of that of S onto Q.

2. If S' is a structure derived from S, and Q a substructure of S' which is canonically isomorphic to S, then every automorphism of Q may be extended uniquely to an inner automorphism of S', relative to the derivation of S' from S.

3. If a structure S contains a substructure T, and if S is canonically reconstructible from T, then every automorphism of T may be extended uniquely to an inner automorphism of S, relative to the derivation of S from T.

4. Given a structure S and a structure S' derived from S, if S is canonically reconstructible from S', then every automorphism of S is inner.

He then applies these results to give a proof of the theorem of Hölder to the effect that every automorphism of the symmetric group on n elements is inner if n is finite and different from six. The proof extends to the case where n is infinite. He states without proof that by the same techniques an extension of another theorem of Hölder can be obtained, namely that if G is any subgroup of the symmetric group on n letters S_n (where n is not equal to 3 or 6), then any automorphism of G can be extended uniquely to an inner automorphism of S_n.

The author announces proofs by these techniques of the theorem of Fine and Schweigert that every automorphism of the group of homeomorphisms of the unit interval onto itself is inner, and of a similar theorem for the disc. These are to appear in later papers in the series. *O. Frink.*

Krull, Wolfgang. Über geordnete Gruppen von reellen Funktionen. Math. Z. **64** (1955), 10–40 (1956).

Let M be a set, Φ a filter on M, F a subgroup of the additive group of all real functions defined on M, $F(\Phi)$ the subgroup of F consisting of all functions vanishing in a set of Φ, $\tilde{F}_\Phi=F/F(\Phi)$; natural examples of such groups arise in integration theory and in the theory of divisibility. $\tilde{F}_\Phi$ is naturally (partially) ordered, the "cone" of elements $\geqq 0$ being the set O_Φ of classes having a representative which is $\geqq 0$ on a set of Φ. The author is interested in obtaining O_Φ as an intersection of "maximal cones", each of which corresponds to a total ordering on $\tilde{F}_\Phi$; he shows first that this can be achieved by considering the sets O_Ψ, where Ψ is an ultrafilter containing Φ; this leads him to a study of the case in which $\Phi=\Psi$ is an ultrafilter, and of the corresponding total ordering, which, as expected, exhibits rather pathological features. Simpler methods are available when F consists for instance of all continuous (or Lebesgue measurable and bounded) functions in an interval. Under restrictions of denumerability, it is possible to obtain O_Φ as intersection of maximal cones O_Ψ such that the total orders corresponding to these maximal cones are all isomorphic. Some conditions are also given under which $\tilde{F}_\Phi$ is a σ-complete lattice. Finally, the last section deals with a completely different question: the author shows that if A is a commutative integrity domain different from its field of quotients, then for any infinite set M, the (complete) product A^M is not a free A-module; this generalizes a theorem proved by R. Baer when A is the ring of integers [Duke Math. J. **3** (1937), 68–122, p. 121], but instead of citing Baer's paper, the author cites an exercise in Bourbaki.

J. Dieudonné (Evanston, Ill.).

Ono, Takashi. Arithmetic of orthogonal groups. II. Nagoya Math. J. **9** (1955), 129–146.

This paper represents an extension of results, called theorems of Hasse type, obtained by the author in a previous paper [J. Math. Soc. Japan **7** (1955), 79–91; MR **16**, 1087]. Let K be a field with characteristic $\neq 2$, V a finite-dimensional vector space over K and f a (non-degenerate, symmetric, bilinear) form on V. Denote by $\Gamma(V, f)$ any one of the following three linear groups on V: the orthogonal group $O(V, f)$, the rotation group $O^+(V, f)$ or the commutator subgroup $\Omega(V, f)$ of $O(V, f)$. Now consider a second vector space W over K and g a form on W. Let Θ be a semi-linear injection (1-1 into) of W into V, with associated automorphism θ of K, such that $g=\lambda\Theta f$ where $\lambda \in K$ and $(\Theta f)(x, y)=f(\Theta x, \Theta y)^{\theta-1}$. Let f_1 be the form f restricted to $\Theta(W)$ and for $\sigma \in O(\Theta(W), f_1)$, define $\Theta\sigma=\Theta^{-1}\sigma\Theta$. Then $\sigma\to\Theta\sigma$ is a group isomorphism, $\Theta(\Gamma(\Theta(W), f_1))=\Gamma(W, g)$ and the group $\Gamma(W, g)$ is said to be "semi-linearly embedded" in $\Gamma(V, f)$. If Θ is linear, the embedding is called linear instead of semi-linear. The form f is said to represent semi-similarly the form g. The author studies this relationship between forms in general and, when K is either a field of algebraic numbers or a field of algebraic functions of one variable over a finite field of characteristic $\neq 2$, obtains theorems of Hasse type. Let $K_\mathfrak{p}$ be the $\mathfrak{p}$-adic completion of K with respect to a place $\mathfrak{p}$ in K and denote by $V_\mathfrak{p}$ the scalar extension of V with respect to $K_\mathfrak{p}$. The following two results indicate the type of theorem obtained: $\Gamma(W, g)$ is linearly embedded in $\Gamma(V, f)$ if and only if $\Gamma(W_\mathfrak{p}, g)$ is linearly embedded in $\Gamma(V_\mathfrak{p}, f)$ for every place $\mathfrak{p}$ in K. (2) If $\Gamma(W_\mathfrak{p}, g)$ is semi-linearly embedded in $\Gamma(V_\mathfrak{p}, f)$ for every place $\mathfrak{p}$ in K, then $\Gamma(W, g)$ is linearly embedded in $\Gamma(V, f)$.

C. E. Rickart (New Haven, Conn.).

Walter, John H. Isomorphisms between projective unitary groups. Amer. J. Math. **77** (1955), 805–844.

Let E be a right vector space of dimension n over a sfield K (characteristic $\neq 2$) and assume given an involution $J:\varphi\to\varphi^J$ in K. A mapping $f:(x, y)\to f(x, y)$ of $E\times E$ into K is called a sesquilinear form if it is additive in both x and y and $f(x\alpha, y\beta)=\alpha^J f(x, y)\beta$ for $\alpha, b \in K$. It is called hermitian if $f(y, x)^J=f(x, y)$ and non-degenerate if $f(x, y)=0$ for all y implies $x=0$. In all that follows f is assumed to be a non-degenerate, hermitian sesquilinear form. The unitary group $U_n(K, f)$ consists of all linear transformations of E into itself which leave f invariant. The inner automorphism of K by $\lambda \in K$ is denoted by λ^*, so $\varphi^{\lambda^*}=\lambda^{-1}\varphi\lambda$. The transformation h_λ: $x\to x\lambda$ in E is semi-linear with respect to the automorphism λ^* of K and is called a homothetic transformation. The group of homothetic transformations is denoted by $H_n(K)$ and the factor group $U_n(K, f)/H_n(K)\cap U_n(K, f)$ is the

projective unitary group $PU_n(K, f)$. The problem of finding the isomorphisms of these groups is the author's concern here.

The problem has been considered by Dieudonné [Mem. Amer. Math. Soc. no. **2** (1951); MR **13**, 531] whose results essentially cover the case in which K is a finite field. The problem for the general unitary groups $U_n(K, f)$ was considered by the reviewer [Amer. J. Math. **73** (1951), 697–716; Bull. Amer. Math. Soc. **57** (1951), 435–448; MR **13**, 532]. The present paper treats the general case of $PU_n(K, f)$. The method used is an extension of one introduced by Mackey [Ann. of Math. (2) **43** (1942), 244–260; MR **4**, 12] and developed by the reviewer [loc. cit.]. The crux of the problem is to obtain a group-theoretic characterization of certain involutions in $PU_n(K, f)$ called extremal. Here an involution is an element of $PU_n(K, f)$ whose square is the identity. More precisely, it is a coset u^* in $PU_n(K, f)$ determined by an element $u \in U_n(K, f)$ such that $u^2=h_\gamma$ with γ in the center Z of K. The element u is called a projective involution. The projective involutions are of four kinds according as (1) $\gamma=\lambda^2$, $\lambda \in Z$, $\lambda^J\lambda=1$, (2) $\gamma=\lambda^2$, $\lambda \in Z$, $\lambda^J\lambda=-1$, (3) $\gamma=\lambda^2$, $\lambda \notin Z$, (4) γ is not a square in K. If u is a projective involution of the first kind, then $\lambda^{-1}u$ is an involution in $U_n(K, f)$, so $E=U^+\oplus U^-$, where $\lambda^{-1}u$ is equal to the identity on U^+, minus the identity on U^- and the subspaces U^+, U^- are non-isotropic. If either U^+ or U^- is one-dimensional, then u^* is called an extremal involution. It is well-known that once the extremal involutions are characterized in the group $PU_n(K, f)$, then the way is clear for obtaining the isomorphisms of these groups. The author succeeds in obtaining such a characterization for $n>4$. *C. E. Rickart* (New Haven, Conn.).

Littlewood, D. E. The Kronecker product of symmetric group representations. J. London Math. Soc. **31** (1956), 89–93.

The study of the reduction of the inner Kronecker product $[\lambda]\times[\mu]$ of two irreducible representations of the symmetric group S_n was first undertaken by Murnaghan [Amer. J. Math. **60** (1938), 761–784]. Recently, Robinson and Taulbee [Proc. Nat. Acad. Sci. U.S.A. **40** (1954), 723–726; MR **16**, 110] gave a general method for obtaining this reduction, and here the author further develops their line of thought. The method rests on the shift of emphasis indicated in the equation

$$((\alpha)\uparrow G)\times(\beta)=((\beta)G\downarrow H\times(\alpha))\uparrow G,$$

where the upward arrow indicates inducing and the downward arrow restricting. The analysis of any inner product $[\lambda]\times[\mu]$ is surprisingly complicated, whether we take a general (α) or the identity representation of a suitably chosen subgroup H of S_n [cf. Robinson and Taulbee, loc. cit.]. The author uses the language of Schur functions and denotes the analogous product by $\{\lambda\}\circ\{\mu\}$.
G. de B. Robinson (Toronto, Ont.).

Murnaghan, Francis D. On the irreducible representations of the symmetric group. Proc. Nat. Acad. Sci. U.S.A. **41** (1955), 1096–1103.

This paper continues recent work of the author [same Proc. **41** (1955), 514–515, 515–518; MR **17**, 12]. Classifying representations of S_n according to the number of rows of the Young diagram, he sets out six rules for determining the reduction of an inner Kronecker product (cf. the preceding review). From the relation

$$(\Gamma\otimes\{\mu\}\times(\Gamma\otimes\{\mu'\})=\Gamma\otimes\{\mu\}\{\mu'\}$$

is derived the reduction of $\Gamma\otimes\{\mu\}$ for any hook representation $\Gamma=(n-1, 1)$ and all partitions of 5 and 6; the cases 2, 3, 4 were considered previously. Proofs are omitted, and an abbreviated notation including the use of the notion of 'division' make for difficult reading.
G. de B. Robinson (Toronto, Ont.).

Hauptman, H.; and Karle, J. Structure invariants and seminvariants for non-centrosymmetric space groups. Acta Cryst. **9** (1956), 45–55.

Borevič, Z. I. On homology theory in groups with operators. Dokl. Akad. Nauk SSSR (N.S.) **104** (1955), 5–8. (Russian)

Let G be a group and $\xrightarrow{\partial}\phi_n\xrightarrow{\partial}\phi_{n-1}\xrightarrow{\partial}\cdots$ be an exact sequence of G-free G-modules ϕ. Let A be an arbitrary abelian group operated on by G. For any G-module ϕ, Hom (ϕ, A) will denote the group of additive homomorphisms $h:\phi\to A$ with G operating on it by the prescription $(gh)(x)=g(h(g^{-1}x))$. It is known that for any G-free G-module ϕ, $H^n(G, \mathrm{Hom}(\phi, A))=0$ for all $n\geqq 0$. This fact along with the exactness of

$$0\to \mathrm{Hom}\,(\partial\phi_j, A)\to \mathrm{Hom}\,(\phi_j, A)\to \mathrm{Hom}(\partial\phi_{j+1}, A)\to 0$$

(which results from the freeness of the ϕ_j and the exactness of the ϕ-sequence) yields

$$H^{n+k}(G, \mathrm{Hom}\,(\partial\phi_j, A))\cong H^n(G, \mathrm{Hom}\,(\partial\phi_{j+k}, A)).$$

Using this observation the author infers that for any sequence ϕ_j like the above which contains a ϕ_0 such that $\phi_0 \bmod \partial\phi_1\cong J$ (=additive groups of integers), $H^{n+k}(G, A)\cong H^n(G, \mathrm{Hom}\,(\partial\phi_k, A))$ and similarly $H_{n+k}(G, A)\cong H_n(G, A\otimes\partial\phi_k)$, where $n\geqq 1$, $k\geqq 0$. For finite groups however it is inferred from the canonical chain-cochain complex and its dual, that the above statement holds for arbitrary n, k and in addition

$$H_n(G, A)\cong H_{n+k}(G, \mathrm{Hom}\,(\partial\phi_k, A)) \text{ and } H^n(G, A)\cong H^{n+k}(G, A\otimes\partial\phi_k).$$

These theorems yield easily the Eilenberg-MacLane cup-product reduction theorem, a similar theorem for finite groups due to the author, and universal coefficient theorems for groups and in particular for finite groups, e.g. for any finite group G operating trivially on an abelian A

$$H_n(G, A)\cong \mathrm{Hom}\,(H^n(G, J), A)+\mathrm{Ext}(H^{n+1}(G, J), A)$$

and

$$H^n(G, A)\cong A\otimes H^n(G, J)+\mathrm{Tor}(A, H^{n+1}(G, J)).$$

W. T. van Est (Utrecht).

Aubert, Karl Egil. Un théorème de représentation dans la théorie des idéaux. C. R. Acad. Sci. Paris **242** (1956), 320–322.

Extending to a non-commutative semigroup the notion of r-ideal of Lorenzen [Math. Z. **45** (1939), 533–553; MR **1**, 101], the author defines a (finite) left x-system in a semigroup D to be a mapping $\mathfrak{a}\to\mathfrak{a}_x$ of the set of (finite) subsets of D into the set of all subsets of D such that (1) $\mathfrak{a}\subseteq\mathfrak{a}_x$, (2) $\mathfrak{a}\subseteq\mathfrak{b}_x$ implies $\mathfrak{a}_x\subseteq\mathfrak{b}_x$, (3) $\mathfrak{a}\mathfrak{b}_x\subseteq\mathfrak{b}_x\frown(\mathfrak{a}\mathfrak{b})_x$. The author discussed these ideal systems in a previous paper [C. R. Acad. Sci. Paris **238** (1954), 2214–2216; MR **15**, 848]. [The author does not mention Lorenzen's definition of an r-system in a non-commutative semigroup, Math. Z. **52** (1949), 483–526; MR **11**, 497.] Let L be a semilattice-ordered semigroup ("gerbier"), and let L be left quasi-integral ($ab\leqq b$ for all a, b in L). Then L is isomorphic with

a finite left x-system in a semigroup D; in fact we need only take $D=L$ and take for the x-system that of all the finitely generated lattice ideals of L.

A. H. Clifford (New Orleans, La.).

Thierrin, Gabriel. Contribution à la théorie des équivalences dans les demi-groupes. Bull. Soc. Math. France **83** (1955), 103–159.

This paper contains proofs of results reported by the author in six earlier communications [C. R. Acad. Sci. Paris **234** (1952), 1519–1521, 1595–1597; **236** (1953), 565–567, 1399–1401, 1723–1725; Acad. Roy. Belg. Bull. Cl. Sci. (5) **39** (1953), 942–947; MR **13**, 902; **14**, 616, 842, 944; **15**, 680]. These will be referred to as [1]-[6], resp. The first two sections of Chapter I are slight expansions of [1] and [2], resp. The third section of Chapter I is new (see below). The three sections of Chapter II are abstracted in [4], [5], and [3], resp. The arrangement of the material in [3] has been greatly improved; the relation denoted by $R_{[H]}$ in [3] is now called ϱ_H, while $R_{(H)}$ is not introduced in the new treatment. Chapter III is abstracted in [6]. The reviews of these six papers give an adequate idea of the contents of the present paper, except for Section 3 of Chapter I.

This new section is devoted to "resorbent homogroups". A homogroup H (=semigroup containing a kernel N which is a group) is called resorbent if (1) H is a union of groups, and (2) the product of any two distinct idempotents is the identity element e of N. The author shows that each right regular equivalence relation R in H can be described in terms of a resorbent subhomogroup K of H containing e, as follows: aRb if and only if $Ka=Kb$. Call this relation φ_K. Then the center Z of H is a resorbent subhomogroup of H; φ_Z is regular on both sides; and the homogroup of all interior endomorphisms $x\rightarrow axa^{-1}$ of H is isomorphic with H/φ_Z. [Note by reviewer. Resorbent homogroups fall into the class of semigroups which are unions of groups, and in which any two idempotents commute. The structure of these is explicitly known; cf. Clifford, [Ann. of Math. (2) **42** (1941), 1037–1049, Th. 3, p. 1044; MR **3**, 199.] For resorbent homogroups, the semilattice P of idempotents is particularly simple. Had the author availed himself of this theorem, the twelve pages of this section would have been reduced to two or three.] *A. H. Clifford* (New Orleans, La.).

Lesieur, L. Sur les demi-groupes réticulés satisfaisant à une condition de chaîne. Bull. Soc. Math. France **83** (1955), 161–193.

The objective of this paper is to formulate, in a residuated lattice-ordered semigroup T, the E. Noether theory of decomposition of ideals of a ring or semigroup into the intersection of primary ideals. Further assumptions must be made on T, and there are many ways of doing so in order that the usual existence and uniqueness theorems hold. The first such theory was given by W. Krull [S.-B. Phys.-Med. Soz. Erlangen **56** (1924), 47–63], and the second by M. Ward and R. P. Dilworth [Trans. Amer. Math. Soc. **45** (1939), 335–354]. In the present paper, the basic assumptions on T are: (1) T is quasi-integral ($ab\leq a$ and $ab\leq b$ for all a, b in T); (2) T has a universal element u ($a\leq u$ for all a in T); (3) T satisfies either the ascending chain condition or the weak descending chain condition. The conjunction of these is called Condition H.

In the early part of the paper, no further conditions are imposed on T beyond Condition H. An element a of T is called primal if a is not the intersection of right residuals of a, both properly containing a. Each element a of T admits at least one, and at most a finite number, of maximal (hence prime) left residuals, and a is right primal if and only if it admits exactly one such. Each element of T can be expressed as the intersection of a finite set of primal elements which can be specialized in such a way that any two such sets can be paired off, with corresponding primals having the same maximal left residuals. Under the ascending chain condition, the existence of a decomposition into $\cap$-irreducible elements is evident; under the weak descending chain condition it follows if we assume that T is weakly $\cap$-continuous: $x\cap \bigcup_\alpha y_\alpha = \bigcup_\alpha (x\cap y_\alpha)$ for every chain $\{y_\alpha\}$ bounded from below.

An element q of T is called right primary if $xy\leq q$ and $y\nleq q$ imply $x^k\leq q$ for some k. Every right primary element is also right primal. q is right primary if and only if q has only one proper prime left residual. Three sets of conditions (I, II, III) on T are given (too complicated to state here) under which decomposition into primaries holds. Assuming I or II, which closely resemble those given by Ward and Dilworth, it is shown that each $\cap$-irreducible element is primary. Assuming III, the proof proceeds from the decomposition into primal elements. Uniqueness theorems are given in each case. The paper closes with a section giving some sufficient conditions under each of which the weak descending chain condition implies the ascending chain condition, e.g. any one of the following: (1) T semi-modular, (2) T weakly $\cap$-continuous, (3) condition II above, (4) condition III above.

A. H. Clifford (New Orleans, La.).

Yamada, Miyuki. On the greatest semilattice decomposition of a semigroup. Kōdai Math. Sem. Rep. **7** (1955), 59–62.

By a semilattice decomposition of a semigroup S is meant the partition of S into equivalence classes of a congruence relation ϱ of S such that S/ϱ is a semilattice (commutative, idempotent semigroup). If σ is another such, we write $\varrho\geqq\sigma$ if $a\varrho b$ implies $a\sigma b$. The set of all such congruence relations on S is a complete semilattice under $\geqq$, and hence there exists a greatest one ϱ. For commutative S, φ was described constructively by Tamura and Kimura [same Rep. **1954**, 109–112; MR **16**, 670]. In the present paper φ is described for non-commutative S as follows. Call a subsemigroup T of S a P-subsemigroup of S if, whenever T contains a word formed from a set of elements of S, then T contains every word formed from this set (without omissions). Then $a\varphi b$ if and only if $xay\in T\leftrightarrow xby\in T$ for every P-semigroup T of S. This congruence, for a single arbitrary set T, was introduced by R. S. Pierce [Ann. of Math. (2) **59** (1954), 287–291; MR **15**, 930].

The reviewer would like to propose an alternative description of φ. To save circumlocution, let S^* be S with an identity element adjoined. Define $a\psi' b$ to mean that there exist elements a, b of S, elements x, y of S^*, and positive integers m, n, such that $a=xcy$, $b=xdy$, and $c^m=d^n$. Let ψ be the transitive closure of ψ'. Then ψ is the greatest congruence relation on S such that S/ψ is a band (idempotent semigroup). S/ψ is a semilattice Ψ of rectangular bands [Clifford, Proc. Amer. Math. Soc. **5** (1954), 499–504, Theorem 3; MR **15**, 930], and the obvious homomorphism of S onto Ψ gives φ.

A. H. Clifford (New Orleans, La.).

Yamada, Miyuki. A note on middle unitary semigroups. Kōdai Math. Sem. Rep. **7** (1955), 49–52.

Let M be the set of all middle units of a semigroup S ($u \in M$ if and only if $aub=ab$ for all $a, b \in S$). S is called M-inversible if, given any $x \in S$, there exist x' and x'' in S such that $x'x$ and $xx'' \in M$, then there exists x^* such that $xx^*=x^*x \in M$. Let us say that a semigroup S is an inflation [reviewer's term] of a subsemigroup T of S if $S^2 \subseteq T$ and there exists an idempotent homomorphism of S onto T. The structure of M-inversible semigroups is completely elucidated by the following theorem. S is M-inversible if and only if it is an inflation of a subsemigroup isomorphic with the direct product of a group and a rectangular band (semigroup $R \times L$ of all ordered pairs (i, j), $i \in R$, $j \in L$, with $(i_1, j_1)(i_2, j_2)=(i_1, j_2)$). [Part of this theorem was found independently by G. Thierrin, Acad. Roy. Belg. Bull. Cl. Sci. (5) **41** (1955), 83–92, cf. Theorem 8; MR **17**, 10.] It is easy to see that a semigroup is M-inversible if and only if it is "inversed" and "rectangular" in the sense of Thierrin.] In the second part of the paper, the author considers what he calls special middle unitary semigroups, namely semigroups in which every idempotent is a middle unit. Every such semigroup is an extension of a nonpotent semigroup (one containing no idempotent) by an M-inversible semigroup. *A. H. Clifford.*

See also: Amitsur, p. 577.

THEORY OF NUMBERS

Kraitchik, M. Les carrés magiques d'ordre 4. Mathesis **64** (1955), 97–115.

The author analyzes the methods of constructing magic squares of order four, and classifies Frenicle's 880 squares into thirteen types by means of the symmetries they exhibit. *R. J. Walker* (Ithaca, N.Y.).

Storchi, Edoardo. Un metodo per la fattorizzazione dei numeri della forma $a_n \pm 1$. Ist. Lombardo Sci. Lett. Rend. Cl. Sci. Mat. Nat. (3) **19(88)** (1955), 405–441.

The author discusses a method of discovering small primitive factors p of a^n-1 by means of various residuacity criteria. The basis of the method is the assumption that p is represented in the form $x^{2k}+ay^{2k}$. From this it follows at once that $-a$ is a $2k$th power residue of p. Supposing that $p=4km+1$, one can ask under what conditions is a a $4k$th power residue of p? The author proves for example that if x and a are both odd then

$$a^{(p-1)/(4k)} \equiv (a/x) \pmod p,$$

where the symbol is that of Jacobi. For example, one has the following criterion of the biquadratic residuacity of 3. Let $p=12n+1=c^2+3d^2$, then 3 is a biquadratic residue of p if and only if c is of the form $12m \pm 1$. This may be compared with the following alternative criterion. Let $p=12n+1=x^2+4y^2$, then 3 is a biquadratic residue of p if and only if 3 divides x. There are as many as 29 theorems based on the above general idea. One having to do with Mersenne primes is as follows. Let both p and $8p+1$ be primes, the latter being of the form f^4+8g^4 with $f=8h\pm3$. Then 2^p-1 is divisible by $8p+1$. For example, $2^{2411}-1$ is divisible by $19289=3^4+8\cdot7^4$. *D. H. Lehmer.*

Duparc, H. J. A. Periodicity properties of certain sets of integers. Nederl. Akad. Wetensch. Proc. Ser. A. **58**= Indag. Math. **17** (1955), 449–458.

The purpose of this paper is to discuss the periodicities of two sequences of integers u_n and v_n defined below. The intent appears to be to produce a sequence with simple rules of generation having quite large periods, possibly with a view of generating pseudo-random digits cheaply. Periodicity follows from recursiveness and boundedness.

Let m be a positive integer. For interesting results m should be ≥ 2, and for applications m is the base of the digital number system. Let k be an integer and let

$$\phi_1(x)=x^s+p_1x^{s-1}+\cdots,\ \phi_2(x)=mx^t+q_1x^{t-1}+\cdots$$

be two polynomials with integer coefficients and such that $\phi_1(0)\neq0$ and $\phi_2(0)\neq0$ with all the roots of ϕ_2 inside the unit circle. Let E be the familiar operator $Eu_n=u_{n+1}$. Let the operators U and V be defined by $U=\phi_1(E)$, $V=\phi_2(E)$. Then the author considers two sequences of integers u_n, v_n ($n=p, 1, \cdots$) such that

$$(1) \qquad Uu_n+Vv_n=k$$

with the condition that u_n is a single digit, that is $0\leq u_n \leq m-1$. The condition (1) determines the u's and the v's uniquely. More or less explicit results about the period are obtained. In typical cases the exponent of m modulo M, the resultant of ϕ_1 and ϕ_2, is involved. *D. H. Lehmer.*

Subba Rao, K. A note on the recurring period of the reciprocal of an odd number. Amer. Math. Monthly **62** (1955), 484–487.

A number of recent notes about the decimal expansion of $1/p$, where p is an odd prime, are discussed and in some cases generalized to the case of $1/n$ where $(n, 10)=1$. The author notes that 10 is a primitive root of $n=7^k$ and so in these cases $1/n$ has a maximum period length of $\phi(n)$. The connection between the reciprocal of the prime $109=10^2+10-1$ and the Fibonacci numbers is extended to other recurring series of the same scale of relation. It is conjectured that infinitely many primes have 10 for a primitive root. *D. H. Lehmer* (Berkeley, Calif.).

Jarden, Dov; and Katz, Alexander. A second-order slowly increasing recurring sequence. Riveon Lematematika **9** (1955), 72. (Hebrew)

The values of $U_n=(\alpha^n-\beta^n)/(\alpha-\beta)$, where α, β are the complex roots of $x^2+x+2=0$, are given for $n=0(1)100$ together with their complete factorization into primes. $|U_n|=1$ for $n=1, 2, 3, 5, 13$. Thus U_{65}, which is divisible by U_5 and U_{13}, is actually a prime number -335257649. The so called characteristic prime factors are shown by underlining. There are no such factors for $n=8$, 12, 18 and 30. *D. H. Lehmer* (Berkeley, Calif.).

Lah, I. Eine neue Art von Zahlen, ihre Eigenschaften und Anwendung in der mathematischen Statistik. Mitteilungsbl. Math. Statist. **7** (1955), 203–212.

The numbers of the title are those appearing in the reciprocal relations: $(-x)_n=\sum S_n{}^k(x)_k$, $(x)_n=\sum S_n{}^k(-x)_k$ with $(x)_n=x(x-1)\cdots(x-n+1)$. It is shown that

$$S_n{}^k=(-1)^n\frac{n!}{k!}\binom{n-1}{k-1},$$

that the reciprocity above holds for general A_n and a_n, and that the numbers are expressible by Stirling numbers of both kinds. A table is given for $k=1(1)n$, $n=1(1)8$. [Reviewer's note: the positive numbers $(-1)^nS_n{}^k=\sigma_{nk}$ have

the double generating function $\sum \sigma_{nk}x^k t^n/n! = \exp xt(1-t)$ and hence are related to the coefficients of Laguerre polynomials; in fact

$$\sigma_n(x) = \sum \sigma_{nk}x^k = L_n(-x) - nL_{n-1}(-x) = xL_n'(-x)/n = xe^{-x}D^n(x^{n-1}e^x)$$

with $D = d/dx$.] *J. Riordan* (New York, N.Y.).

Carlitz, L. A degenerate Staudt-Clausen theorem. Arch. Math. 7 (1956), 28–33.

The author proves three theorems of which the first resembles the Staudt-Clausen theorem. Put

$$\frac{x}{(1+\lambda x)^\mu - 1} = \sum_{m=0}^{\infty} \beta_m(\lambda)\frac{x^m}{m!} \quad (\lambda\mu = 1),$$

so that $\beta_m(\lambda)$ is a polynomial in λ with rational coefficients. Theorem 1. Let $\lambda = a/b$, $(a, b) = 1$. Then for m even

$$\beta_m(\lambda) = A_m - \sum_{(p-1)|m,\, p\nmid a} \frac{1}{p}$$

where A_m is a rational number whose denominator contains only primes occurring in b. For m odd $\beta_1(\lambda) = \frac{1}{2}(\lambda - 1)$ and $\beta_m(\lambda) = A_m - \frac{1}{2}$ $(m>1)$ provided $2|a$, $4\nmid a$ while if $2\nmid a$ or $4|a$, then $\beta_m(\lambda) = A_m$. In particular, when λ is a rational integer then A_m is also an integer. [For the proof cf. N. E. Nörlund, Vorlesungen über Differenzenrechnung, Springer, Berlin, 1924, p. 32.] Theorems 2 and 3 are concerned with congruence properties of the polynomials $\beta_m(\lambda)$.
A. L. Whiteman (Los Angeles, Calif.).

Hartman, S. Sur un type de lacunarité. Matematiche, Catania 10 (1955), 57–61.

An infinite increasing sequence of natural numbers is said to satisfy condition (P) if there exists a natural number K such that, for every positive integer l, can be found l consecutive members of the sequence, say n_{k+1}, n_{k+2}, $\cdots$, n_{k+l}, satisfying $n_{k+i+1} - n_{k+i} < K$ $(i = 1, 2, \cdots, l-1)$. The author observes that those infinite increasing sequences which do not possess property (P) may be thought of as subject to a new type of "gap" condition, and, after giving some simple examples of sequences possessing property (P), he constructs a sequence of positive density which does not satisfy (P). He conjectures that the sequence of primes also does not satisfy (P) and in a footnote he points out that his conjecture may readily be confirmed by means of a certain result of Hardy and Littlewood. In conclusion the following theorem is proved: If $\sigma_1, \sigma_2, \cdots, \sigma_\mu$ $(\mu < \infty)$ are infinite increasing sequences which do not satisfy (P), then neither does the increasing sequence composed of all terms which appear in any of the sequences $\sigma_1, \cdots, \sigma_\mu$.
H. Halberstam (Berkeley, Calif.).

Sierpinski, W. Sur la lacunarité au sens de S. Hartman de la suite de tous les nombres premiers. Matematiche, Catania 10 (1955), 67–70.

In the notation of the preceding review, the author gives an independent proof that the sequence of primes obeys Hartman's gap condition. He does so by establishing the following theorem: For every $\varepsilon > 0$ there exists a number $\mu = \mu(\varepsilon)$ such that $\pi(n+k) - \pi(n) < k\varepsilon$ for $k > \mu$, $n = 1, 2, \cdots$. The proof is elementary, and the result leads easily to the conclusion that the sequence of primes does not satisfy condition (P). *H. Halberstam.*

Erdös, P.; and Fuchs, W. H. J. On a problem of additive number theory. J. London Math. Soc. 31 (1956), 67–73.

If $a_1, a_2, \cdots$ is an infinite sequence of integers such that $0 \leq a_1 \leq a_2 \cdots$, denote by $f(n)$ and $r(n)$, respectively, the number of solutions of the equation $a_i + a_j = n$ and of the inequality $a_i + a_j \leq n$. Erdös and Turán [same J. 16 (1941), 212–215; MR 3, 270] conjectured that (1) $r(n) = cn + O(1)$, c constant, cannot hold, and that if $f(n) > 0$, $n > n_0$, then $\limsup f(n) = \infty$. G. A. Dirac [ibid. 26 (1951), 312–313; MR 13, 326] pointed out that the answer to (2) depended on how the solutions with $i = j$ were to be counted, and proved in two out of the three cases that $f(n)$ cannot be constant for all $n > n_0$. In the present paper it is proved that (a) $r(n) = cn + o(n^{\frac{1}{4}} \log^{-\frac{1}{2}} n)$ cannot hold with $c > 0$, and that (b) if $c \geq 0$, $a_k < Ak^2$, then

$$\limsup \{n^{-1}\textstyle\sum_{k=0}^{n} (f(k) - c)^2\} > 0$$

in all three cases. The proof uses the generating function $\phi(z) = \sum f(n)z^n$ and an inequality for the integral of $|\phi(z)|^2$ along an arc of the circle $|z| = r < 1$, combined with skillful manipulation of suitable other inequalities.
R. D. James (Vancouver, B.C.).

Moessner, Alfredo. Problemi diofantei. Boll. Un. Mat. Ital. (3) 10 (1955), 574–576.

Georgiev, G. On the solution in rational numbers of certain Diophantic equations. Prace Mat. 1 (1955), 201–238. (Polish. Russian and English summaries)

An exhaustive elaboration of the method expounded more briefly in Uspehi Mat. Nauk (N.S.) 8 (1953), no. 2(54), 115–118; MR 14, 1063. *J. W. S. Cassels.*

Sastry, S. On some systems of Diophantine equations. J. Sci. Res. Banaras Hindu Univ. 5 (1955), no. 2, 1–6.

The author shows how a representation of an integer r^2 in the form $a^2 + ab + b^2$ can be used to obtain integer solutions of the system of equations

$$x_1^j + \cdots + x_i^j = y_1^j + \cdots + y_8^j \quad (1 \leq j \leq 6).$$

W. H. Simons (Vancouver, B.C.).

Dutta Mishra, Shiva. On Prouhet-Lehmer problem. J. Sci. Res. Banaras Hindu Univ. 5 (1955), no. 2, 7–10.

Let $P(k, s)$ be the least value of j such that the $k(s-1)$ Diophantine equations

$$\sum_{i=1}^{j} x_{i1}^h = \sum_{i=1}^{j} x_{i2}^h = \cdots = \sum_{i=1}^{j} x_{is}^h \quad (1 \leq h \leq k),$$

have a solution in positive integers. Numerical examples are used to obtain the following bounds for $P(k, s)$: $P(4, 3) \leq 8$, $P(6, 3) \leq 14$, $P(7, 3) \leq 26$, $P(4, 5) \leq 12$, $P(5, 5) \leq 36$. *W. H. Simons* (Vancouver, B.C.).

Prakash Srivastava, Om. On Prouhet-Lehmer problem. J. Sci. Res. Banaras Hindu Univ. 5 (1955), no. 2, 59–62.

$P(k, s)$ being defined as in the previous review, it is shown here that $P(4, q) \leq 2q + 2$. Numerical examples are given to show that $P(4, 5) \leq 10$, $P(4, 6) \leq 12$.
W. H. Simons (Vancouver, B.C.).

Möller, Kurt. Untere Schranke für die Anzahl der Primzahlen, aus denen x, y, z der Fermatschen Gleichung $x^n + y^n = z^n$ bestehen muss. Math. Nachr. 14 (1955), 25–28.

Generalizing a theorem of Lucas, the author shows that in case n is odd and divisible by r distinct primes, the conditions $x^n + y^n = z^n$, $x < y < z$ imply that x contains at least r, and y and z each at least $r+1$ distinct prime factors. The method uses essentially Lucas' function $U_n = (\alpha^n - \beta^n)/(\alpha - \beta)$ and its algebraic factorization when

n is composite. [For previous theorems for $r=1$ see L. E. Dickson, History of the theory of numbers, v. 2, Carnegie Inst. Washington, 1920, pp. 754–761.] *D. H. Lehmer.*

Sexton, Charles R. Count of primes of form $6n\pm1$ up to 400,000. Riveon Lematematika 9 (1955), 73–74.

The number of primes of each of the forms $6n+1$ and $6n-1$ between 10^4n and $10^4(n+1)$ is given for $n=0(1)39$ with certain subtotals. For $n\leqq19$ the results are compared with those obtained by L. Poletti [Tavole di numeri primi, Hoepli, Milano, 1920, pp. 244, 246] and there are 12 discrepancies. The first of these has been independently considered by the reviewer and the author's result is correct. The question of the least value of x for which there are more primes of the form $6n+1$ than $6n-1$ less than x remains unanswered. The data do not prove conclusively that $x>400000$. *D. H. Lehmer.*

Gupta, O. P.; and Luthra, S. Partitions into primes. Proc. Nat. Inst. Sci. India. Part A. 21 (1955), 181–184.

The authors give a table showing the number of partitions of n into prime parts for $n=1(1)300$. The table was computed in two ways, once by a recurrence using the number $P(n, p_r)$ of partitions of n into primes such that p_r is the largest part, and a second time using the function $(1-x^2)(1-x^3)(1-x^5)\cdots$ whose reciprocal generates the number of partitions into primes. The number of partitions of 300 is only 627307270. *D. H. Lehmer.*

Teuffel, E. Beweise für zwei Sätze von H. F. Scherk über Primzahlen. Jber. Deutsch. Math. Verein. 58 (1955), Abt. 1, 43–44.

The following propositions were stated by H. F. Scherk [J. Reine Angew. Math. 10 (1833), 201–208]. Let p_n denote the nth prime, with the convention that $p_1=1$, $p_2=2$, etc. For $n>1$ there exist choices of $\varepsilon_r=\pm1$ and $\varepsilon_r'=\pm1$ such that

$$p_{2n-1}=\sum_{r=1}^{2n-2}\varepsilon_r p_r \text{ and } p_{2n}=2\varepsilon_{2n-1}'p_{2n-1}+\sum_{r=1}^{2n-2}\varepsilon_r' p_r.$$

The writer obtains these (the second in the slightly stronger form with $\varepsilon_{2n-1}'=1$ and $\varepsilon_{2n-2}'=-1$) as corollaries of this result: for $n\geqq2$, any even integer $2k$ satisfying $0\leqq2k\leqq\sum_{r=1}^{2n-1}p_r$ is expressible in the form

$$2k=p_{2n-1}+\sum_{r=1}^{2n-1}\varepsilon_r p_r$$

with suitably chosen $\varepsilon_r=\pm1$. The proof employs Bertrand's "postulate" and induction. *I. Niven.*

Kuhn, P. Eine Verbesserung des Restgliedes beim elementaren Beweis des Primzahlsatzes. Math. Scand. 3 (1955), 75–89.

Using elementary methods, it is shown that the Selberg identity

$$(1)\qquad \theta(x)\log x+\sum_{p\leqq x}\theta\left(\frac{x}{p}\right)\log p=2x\log x+O(x)$$

yields

$$(2)\qquad \theta(x)=x+O(x\log^{-c}x);\ c=1/10.$$

Though it is not difficult to see that the method (which is an adaptation of that of Selberg) will produce some constant $c>0$ for which (2) is a consequence of (1), the calculations necessary for obtaining $c=1/10$ are a bit involved. *H. N. Shapiro* (New York, N.Y.).

Postnikov, A. G.; and Romanov, N. P. A simplification of A. Selberg's elementary proof of the asymptotic law of distribution of prime numbers. Uspehi Mat. Nauk (N.S.) 10 (1955), no. 4(66), 75–87. (Russian)

The underlying idea of this paper is to give an elementary proof of the prime-number theorem in the form

$$(1)\qquad M(x)=\sum_{n\leqq x}\mu(n)=o(x).$$

This deduction is made from the elementary identity

$$(2)\qquad M(x)\log x+\sum_{p\leqq x}M\left(\frac{x}{p}\right)\log p=O(x).$$

This is an old identity which appeared in Landau's thesis, and a direct simple path from (2) to (1) would be rather interesting. However, in addition to (2), the authors use the Selberg identity

$$(3)\qquad \theta(x)\log x+\sum_{p\leqq x}\theta\left(\frac{x}{p}\right)\log p=2x\log x+O(x).$$

This is applied to eliminate the explicit appearance of the primes in (2), converting it into the inequality

$$(4)\qquad |M(x)|\leqq\frac{1}{\log x}\sum_{n\leqq x}\left|M\left(\frac{x}{n}\right)\right|+O\left(\frac{x\log\log x}{\log x}\right).$$

Then applying an iteration scheme analogous to that used by Selberg, (1) is deduced from (4). The only "advantage" gained by dealing with $M(x)$ rather than $\theta(x)$ stems from the fact that $M(x)$ has limited jumps at the integers (since $|\mu(n)|\leqq1$). *H. N. Shapiro.*

Eda, Yoshikazu. On the prime number theorem. Sci. Rep. Kanazawa Univ. 2 (1953), no. 1, 23–33.

The elementary proof of the prime number theorem as given by Selberg is "detranscendentalized" by replacing the function $\log x$ by $\sum_{n\leqq x}1/n$. Though this paper is substantially the same in spirit as that of Fogels [Latvijas PSR Zinātņu Akad. Fiz. Mat. Inst. Raksti 2 (1950), 14–45; MR 13, 824], the details are carried out much more neatly here. *H. N. Shapiro* (New York, N.Y.).

Eda, Yoshikazu. On the Selberg's inequality. Sci. Rep. Kanazawa Univ. 2 (1953), no. 1, 7–13.

Selberg's identity

$$(1)\qquad \theta(x)\log x+\sum_{p\leqq x}\theta\left(\frac{x}{p}\right)\log p=2x\log x+O(x)$$

may be viewed as a translation in terms of primes of

$$(2)\qquad \sum_{n\leqq x}\frac{\mu(n)}{n}\log^2\frac{x}{n}=2\log x+O(1).$$

This last has been generalized by the reviewer to

$$(3)\qquad V_K(x)=\sum_{n\leqq x}\frac{\mu(n)}{n}\log^k\frac{x}{n}$$
$$=k\log^{k-1}x+\sum_{j=1}^{k-2}k_j\log^j x+O(1).$$

In this paper the constants k_j are determined in terms of certain "Euler constants". A generalization of (1) is then obtained by expressing $V_K(x)$ in terms of primes. *H. N. Shapiro* (New York, N.Y.).

Piehler, Joachim. Über die Charaktere quadratischer Formen. Wiss. Z. Martin-Luther-Univ. Halle-Wittenberg. Math.-Nat. Reihe 4 (1955), 1215–1224.

H. J. S. Smith, Minkowski, and, later, G. Pall developed the theory of orders of quadratic forms in n variables. Pall [Quart. J. Math. Oxford Ser. 6 (1935), 30–51] de-

veloped the order invariants systematically and introduced simplifications over the classical results. The author of this paper, using the notation of H. Brandt [Math. Ann. **124** (1952), 334–342; MR **14**, 454], gets Pall's results and goes on to define the generic characters of forms and their concomitants, f_ν, primitive forms the elements of whose matrices are, except for certain factors, the ν-rowed minors of the matrix of f for $\nu=1, 2, \cdots, n-1$. Certain so-called prime discriminant divisors (Primdiskriminanteiler) are defined in terms of the order invariants and it is shown that the primitive forms f_ν possess all and only characters with respect to these prime discriminant divisors. Relationships among characters are found and simultaneous characters are defined. The Hasse and Hilbert symbols are not used. *B. W. Jones.*

Polosuev, A. M. A multidimensional case of unimprovable estimates of trigonometric sums with exponential functions. Dokl. Akad. Nauk SSSR (N.S.) **104** (1955), 186–189. (Russian)

The author generalizes a result of N. M. Korobov [same Dokl. (N.S.) **89** (1953), 597–600; MR **15**, 15] by showing that given any function $\phi(p)$ which tends to infinity as $p\to\infty$, no matter how slowly, real numbers $\alpha_1, \alpha_2, \cdots, \alpha_s$ can be constructed such that

$$\sum_{x=1}^{p} \exp\{2\pi i \sum_{l=1}^{s} m_l\alpha_l q_l^x\}=o\{\varphi(p)\},$$

for any integers $m_1, m_2, \cdots, m_s$ not all zero. Here $q_1, q_2, \cdots, q_s$ are fixed integers greater than unity. It is shown, further, that for no real numbers $\alpha_1, \alpha_2, \cdots, \alpha_s$ can the right-hand side of this result be replaced by $O(1)$. *R. A. Rankin* (Glasgow).

Voelker, Dietrich. Generalization of the fundamental functions of the theory of numbers. Univ. Nac. Tucumán. Rev. Ser. A. **10** (1954), 137–139 (1955). (Spanish. German summary)

The author defines number-theoretic functions which are generalizations of those of Liouville and Möbius, and shows that the generating Dirichlet series are $[\varphi(2s)/\varphi(s)]^k$ and $1/\varphi(s)^k$. *R. Bellman* (Santa Monica, Calif.).

Kubilyus, I. P. Probability methods in number theory. Vestnik Leningrad. Univ. **10** (1955), no. 11, 59–60. (Russian)

A real function $f(m)$, defined on the sequences of all natural numbers, is called strongly additive if (i) $f(mn)=f(m)+f(n)$, $(m, n)=1$, and (ii) $f(p^\alpha)=f(p)$, p prime and $\alpha=2, 3, \cdots$. Set $A_n=\sum_{p\le n} f(p)/p$, $B_n^2=\sum_{p\le n} f^2(p)/p$ and assume that (iii) $B_n\to\infty$ as $n\to\infty$ [see Erdös and Kac, Amer. J. Math. **62** (1940), 738–742; MR **2**, 42; Kac, Bull. Amer. Math. Soc. **55** (1949), 641–665; MR **11**, 161; Halberstam, J. London Math. Soc. **30** (1955), 43–53; **31** (1956), 1–14, 14–27; MR **16**, 569; **17**, 461]. The author states, without proof, the following theorem, of interest in that, to the reviewer's knowledge, it is the first published result of an "if and only if" nature about the distribution of strongly additive functions subject to condition (iii):

Assume that there exists a function $r(n)$ such that (iv) $B_{r(n)}=B_n(1+o(1))$, $\ln r(n)=o(\ln n)$. Then the number of positive integers $m\le n$ satisfying $f(m)<A_n+B_nx$ is equal to $nF(x)+o(n)$ at points of continuity of $F(x)$, where $F(x)$ is some distribution function, if and only if there exists a non-decreasing function $K(u)$ with variation one such that for all $u\ne0$

$$B_n^{-2}\sum_{p\le n,\, f(p)\le B_nu} f^2(p)/p\to K(u),\quad n\to\infty.$$

If these conditions are satisfied, the characteristic function $\phi(t)$ corresponding to $F(x)$ is given by

$$\ln\phi(t)=\int_{-\infty}^{\infty}(e^{itu}-1-itu)u^{-2}dK(u).$$

Also stated is a corollary and a second theorem which, however, was already announced by the author in a preceding note [Dokl. Akad. Nauk SSSR (N.S.) **103** (1955), 361–363; MR **17**, ***].

If $f(p)=O(1)$ it is seen that (iv) is satisfied, that $K(u)=1$ if $u>0$ and 0 if $u<0$, $\ln\phi(t)=-\frac{1}{2}t^2$, and that the corresponding distribution is therefore normal. However, if $f(p)=o(B_n)$ (iv) may no longer be true, and thus the widest class of functions subject to (iii) hitherto discussed (see above ref.) is not covered by the present theorem. *H. Halberstam* (Berkeley, Calif.).

See also: Comét, p. 673.

Analytic Number Theory

Voelker, Dietrich. Sufficient conditions for the validity of the Riemann hypothesis. Univ. Nac. Tucumán. Rev. Ser. A. **10** (1954), 141–149 (1955). (Spanish. German summary)

The author translates the Riemann hypothesis into an equivalent condition concerning the abscissa of convergence of some particular Laplace transforms. *R. Bellman* (Santa Monica, Calif.).

Maass, Hans. Die Bestimmung der Dirichletreihen mit Grössencharakteren zu den Modulformen n-ten Grades. J. Indian Math. Soc. (N.S.) **19** (1955), 1–23.

Generalizing a previous paper [Math. Ann. **122** (1950), 90–108; MR **12**, 319], the author sets up, by the same method, a one-one correspondence between a matrix modular form of degree n and even integral dimension $-k$ and a Dirichlet series. The paper is essentially devoted to establishing the explicit formula

$$\int_{Y>0}e^{-2\pi\operatorname{tr}(TY)}u(Y)|Y|^{s-\frac{1}{2}(n+1)}[dY] =(2\pi)^{-ns}\Gamma(s-\alpha_1)\cdots\Gamma(s-\alpha_n)\pi^{\frac{1}{4}n(n-1)}u_1(E).$$

Here $Y=(y_{\mu\nu})$ is a positive real matrix of order n, $u(Y)$ is a bounded eigensolution of Selberg's differential system

$$\{\operatorname{tr}(Y\partial/\partial Y)^k+\lambda_k\}u(Y)=0\ (k=1, \cdots, n),$$

where $\partial/\partial Y=\{e_{\mu\nu}\partial/\partial y_{\mu\nu}\}$, $e_{\mu\nu}=1$ $(\mu=\nu)$, $\frac{1}{2}$ $(\mu\ne\nu)$, E is the unit matrix, $u_1(y)=u(Y)[S^{-1}]$, $T=SS'>0$, and the α's are determined uniquely by the λ's. *J. Lehner.*

Grosswald, Emil. The average order of an arithmetic function. Duke Math. J. **23** (1956), 41–44.

Let $F(n)$ stand for the number of prime divisors of n, distinct or not, and set $a(n)=2^{F(n)}$. The author's main result is:

$$(1)\quad \sum_{n\le x}a(n)=(r+1)C_1x\log(2^{-r/2}x)-(r+1)C_2x+O(x^\epsilon)$$

where

$$C_1=\tfrac{1}{4}\prod_{p\ge3}(1+(p^2-2p)^{-1}),$$

$$C_2=C_1(2\sum_{p\ge3}\log p/(p-1)(p-2)-2\log2-2\gamma+1),$$

γ is Euler's constant, $r=[\log x/\log 2]$, and the constant c satisfies $c<0.84$. The classical method of evaluating sums of coefficients of Dirichlet series is applied to the function

$$f(s)=\prod_{p\geqq 3}(1-2p^{-s})^{-1}=\sum_{n=1}^{\infty} a_n n^{-s},$$

where $a_n=a(n)$ if n is odd, and $a_n=0$ otherwise. Some difficulties have had to be overcome in the accompanying process of contour integration. The paper concludes with some remarks on the likely true order of magnitude of the error term in (1). *H. Halberstam* (Berkeley, Calif).

See also: Siegel, p. 602; Huber, p. 603.

Geometry of Numbers, Diophantine Approximation

Mahler, Kurt. On compound convex bodies. I. Proc. London Math. Soc. (3) **5** (1955), 358–379.

Let $1\leqq p<n$ and let $X^{(\pi)}=(x_{\pi 1}, \cdots, x_{\pi n})$ $[\pi=1, 2, \cdots, p]$ denote p points in euclidean n-space R_n. They determine the p-vector (1) $\Xi=[X^{(1)}, \cdots, X^{(p)}]$. The $N=\binom{n}{p}$ components $\xi_1, \cdots, \xi_N$ of Ξ are the minors of order p of the matrix $(x_{\pi\nu})$ arranged in some definite order. Ξ can be interpreted as a point on an algebraic manifold $\Omega(n, p)$ in R_N. Let $K^{(1)}, \cdots, K^{(p)}$ denote convex bodies in R_n symmetric with respect to its origin. If each $X^{(\pi)}$ ranges over $K^{(\pi)}$, Ξ will range over a closed bounded set on $\Omega(n, p)$ whose convex hull $\mathrm{K}=[K^{(1)}, \cdots, K^{(\pi)}]$ is a convex body in R_N symmetric with respect to its origin. If each $K^{(\pi)}$ has the distance function $F^{(\pi)}(X)$, that of K will be $\Phi(\mathrm{H})=\inf \sum_\varrho \prod_\pi F^{(\pi)}(X_\varrho^{(\pi)})$, where the lower bound is extended over all finite decompositions $\mathrm{H}=\sum_\varrho [X_\varrho^{(1)}, \cdots, X_\varrho^{(p)}]$. Here H may be any point in R_N.

Put $P=\binom{n-1}{p-1}$ and let $c_1, \cdots$ denote positive constants which depend only on n and p. Let $V(K), \cdots$ denote the volume of $K, \cdots$. In this paragraph let $K^{(1)}=\cdots=K^{(p)}=K$. Theorem 1: There are c_1, c_2 such that $c_1\leqq V(\mathrm{K})V(K)^{-P}\leqq c_2$ for every K. Let $\Delta(K), \cdots$ denote the lattice determinant of $K, \cdots$. Theorem 1 combined with Minkowski's and Hlawka's inequalities $2\zeta(n)\Delta(K)\leqq V(K)\leqq 2^n\Delta(K)$ yields Theorem 2: There are c_4, c_5 such that $c_4\leqq\Delta(\mathrm{K})\Delta(K)^{-P}\leqq c_5$ for every K. — If $X^{(1)}, \cdots, X^{(p)}$ range in dependently over the points of a lattice L in R_n, the points (1) form a set $\Pi\subset\Omega(n, p)$. The finite sums $\sum_\varrho \Xi_\varrho$ with $\Xi_\varrho\subset\Pi$ form a lattice $\Lambda=[L]^{(p)}$ in R_N. Let $0<m_1\leqq m_2\leqq\cdots\leqq m_n$ $[0<\mu_1\leqq\mu_2\leqq\cdots\leqq\mu_N]$ denote the successive minima of K in L [of K in Λ]. Number the N products $M_\varkappa=m_{\nu_1}m_{\nu_2}\cdots m_{\nu_p}$ $[1\leqq\nu_1<\nu_2<\cdots<\nu_p\leqq n]$ in the order of increasing size. Theorem 3: There is a c_7 such that $c_7M_\varkappa\leqq\mu_\varkappa\leqq M_\varkappa$ $[\varkappa=1, \cdots, N]$ for every K and L. — The convex body K^{-1} consists of all those $Y=(y_1, \cdots, y_n)$ for which $|XY|=|\sum_1^n x_\nu y_\nu|\leqq 1$ for all $X=(x_1, \cdots, x_n)\subset K$. Theorem 4: Let $p=n-1$ [thus $\mathrm{K}\subset R_n$]. Then there are c_9, c_{10} such that $c_9V(K)K^{-1}\subset\mathrm{K}\subset c_{10}V(K)K^{-1}$ for every K. The lattice L^{-1} consists of those Y for which XY is an integer for all $X\subset L$. It is similar to $[L]^{(n-1)}$. Let $0\leqq m_1'\leqq\cdots\leqq m_n'$ denote the successive minima of K^{-1} in L^{-1}. Theorems 3 and 4 imply Theorem 5: There are c_{11}, c_{12} such that $c_{11}\leqq m_k m_{n-k+1}'\leqq c_{12}$ $[k=1, \cdots, n]$. A direct proof of this theorem with $c_{11}=1$, $c_{12}=(n!)^2$ was given by Mahler [Časopis Pěst. Mat. Fys. **68** (1939), 93–102; MR **1**, 202].

The lattice L_0 $[\Lambda_0]$ consists of all points in R_n $[R_N]$ with integral coordinates. Let (a_{hk}) be an nth-order square matrix. The elements of the square matrix $(\alpha_{\eta\varkappa}^{(p)})$ of order N are the N^2 minors of order p of (a_{hk}), the ordering of its rows and columns being the same as that of the $\xi_\varkappa$. Let

$$m=\min_{x\in L_0}\ \max_{h=1,\cdots,n}\left|\sum_{k=1}^{n} a_{hk}x_k\right|$$

and

$$\mu=\min_{\Xi\in\Lambda_0}\ \max_{\eta=1,\cdots,N}\left|\sum_{\varkappa=1}^{N}\alpha_{\eta\varkappa}^{(p)}\xi_\varkappa\right|.$$

Theorem 6: There are c_{15}, c_{16} such that $\mu\leqq c_{15}m^{(n-p)/(n-1)}$ and $m\leqq c_{16}\mu^{1/p}$ for all unimodular (a_{hk}). *P. Scherk.*

Mahler, Kurt. On compound convex bodies. II. Proc. London Math. Soc. (3) **5** (1955), 380–384.

[Cf. the preceding review]. Let $p=r+s$; $r>0$, $s>0$. Let $K^{(1)}=K^{(2)}=\cdots=K^{(r)}=K_1$; $K^{(r+1)}=\cdots=K^{(p)}=K_2$; $\mathrm{K}=[K^{(1)}, \cdots, K^{(p)}]$. Put $s=V(\mathrm{K})\{V(K_1)^rV(K_2)^s\}^{-P/p}$. Let $n\geqq 3$; $2\leqq p\leqq n-1$. Then S has no upper bound which depends only on n and p. There is a $c>0$ such that $S\geqq c$ for all K_1, K_2. *P. Scherk* (Saskatoon Sask.).

Mahler, Kurt. On the minima of compound quadratic forms. Czechoslovak Math. J. **5(80)** (1955), 180–193. (Russian summary)

[The second last review is quoted as I]. Let (a_{hk}) be symmetric and let $F(X)=\sum_{h,k=1}^n a_{hk}x_hx_k$ be positive definite. Then $(\alpha_{\eta\varkappa}^{(p)})$ also is symmetric and $\Phi^{(p)}(\Xi)=\sum_{\eta,\varkappa=1}^n \alpha_{\eta\varkappa}^{(p)}\xi_\eta\xi_\varkappa$ is also positive definite. Let $0<m_1\leqq m_2\leqq\cdots\leqq m_n$ $[0<\mu_1^{(p)}\leqq\cdots\leqq\mu_N^{(p)}]$ denote the successive minima of $F(X)$ in L_0 [of $\Phi^{(p)}(\Xi)$ in Λ_0]. Define $M_\varkappa$ as in I. Let G_n denote the unit sphere in R_n. Theorem 1: $\Delta(G_n)^{2P}M_k\leqq\mu_k^{(p)}\leqq M_k$ $[k=1, 2, \cdots, N]$. It implies Theorem 2: $\Delta(G_n)^{2P}m_1^p\leqq\mu_1^{(p)}\leqq\Delta(G_n)^{-2}m_1^{(n-p)/(n-1)}$ if $|a_{hk}|=1$. Theorem 3: $\Delta(G_n)^{2N}\leqq\mu_\eta^{(p)}\mu_{N-\eta+1}^{(n-p)}\leqq\Delta(G_n)^{-2}$ $[\eta=1, 2, \cdots, N]$. The final Theorem 4 shows that Theorem 3 of I holds with $c_7=n^{-p/2}\Delta(G_n)^P$ if $L=L_0$, $\Lambda=\Lambda_0$.

P. Scherk (Saskatoon, Sask.).

Szüsz, P. Lösung eines Problems von Herrn Hartman. Studia Math. **15** (1955), 43–55.

Let $N=N_{\alpha,\beta}(n, I)$ be the number of integers in 1, 2, $\cdots$, n such that the fractional parts $(h\alpha)$, $(h\beta)$ of $h\alpha$ and $h\beta$ are the coordinates of a point in a rectangular subinterval I of the unit square, α and β being numbers in the interval 0 to 1 such that 1, α, and β are linearly independent over the rationals. If $|I|$ is the area of the rectangle I, then N/n approaches $|I|$ as n goes to infinity, but the value of $N-n|I|$ may not be bounded. S. Hartman [Colloq. Math. **1** (1948), 239–240] has asked if $|N-(q\alpha)(q\beta)n|$ is bounded, where q is an integer and I is the rectangle with (0, 0) a corner and sides $(q\alpha)$ and $(q\beta)$, since Ostrowski [Jbr. Deutsch. Math. Verein. **39** (1930), Abt. 1, 34–46] has shown the corresponding result to be true in one dimension. By a construction using continued fractions the author shows this to be false with $q=1$ for any irrational β and any one of an uncountable set of α's corresponding to the given β. In a note added in proof he states that with $(q\alpha)\leqq(q\beta)$ the difference $|N-(q\alpha)n|$ is bounded if I is the parallelogram whose sides are the complex numbers $(q\alpha)/(q\beta)$ and $(q\alpha)+i(q\beta)$.

Marshall Hall, Jr. (Columbus, Ohio).

Korobov, N. M. Numbers with bounded quotient and their applications to questions of Diophantine approximation. Izv. Akad. Nauk SSSR. Ser. Mat. 19 (1955), 361–380. (Russian)

Let q be an arbitrary integer greater than unity, and let

$$\alpha=0\cdot\delta_1\delta_2\delta_3\cdots$$

be the 'decimal' expansion of a real number α in the scale of q. Suppose that, for each $n\geqq 1$, there exists an integer $\lambda=\lambda(n)$ such that among the n-figure numbers

$$\delta_1\delta_2\cdots\delta_n,\ \delta_2\delta_3\cdots\delta_{n+1},\ \cdots,\ \delta_\lambda\delta_{\lambda+1}\cdots\delta_{\lambda+n-1}$$

each of the q^n different n-figure numbers occurs at least once. If there exists a number $C_1=C_1(\alpha)$ such that, for all $n\geqq 1$, $\lambda(n)/q^n<C_1$, the number α is said to be a number with bounded ratio. In the theory of the distribution of the fractional parts of the numbers αq^x $(x=1, 2, \cdots)$, such numbers play a role similar to that played by numbers with bounded partial quotients in the theory of Diophantine approximation. Numbers α of the latter kind have the property that for them, and only for them, does there exist a constant $C=C(\alpha)$ such that the inequalities

$$0<x<Ct,\ |\alpha x-y-\beta|<t,$$

possess integer solutions in x only for arbitrary β and $t\geqq 1$.

The author shows that a necessary and sufficient condition for the existence of a number $C>0$ such that for arbitrary β and $t\geqq 1$ there exist integers x and y satisfying the inequalities

$$0\leqq x<Ct,\ |\alpha q^x-y-\beta|<1/t,$$

is that α is a number of bounded ratio, and he constructs an infinite class of numbers of bounded ratio. He also constructs an infinite class of numbers α for which

$$|\alpha q^x-y-\beta|<(1+\varepsilon)/x$$

has infinitely many solutions in x, y for each β, where ε is an arbitrary positive number. Some of these results were already proved in an earlier paper of the author [Dokl. Akad. Nauk SSSR (N.S.) 89 (1953), 397–400; MR 14, 852] where the generalisations now mentioned were stated without proof.

The concept of bounded ratio is extended to systems of s numbers α_i expressed in scales of bases q_i $(i=1, \cdots, s)$ and similar results are proved. These are applied to questions concerning the distribution of fractional parts, numbers $\alpha_1, \alpha_2, \cdots, \alpha_s$ being constructed such that the system of functions $\alpha_i q_i^x$ $(x=1, 2, \cdots)$ are uniformly distributed in s-dimensional space. It is also shown that the number $N(v)$ of such points lying in a region v of the s-dimensional cube, for $x\leqq P$, satisfies

$$N(v)=vP+O(P^{1-1/(s+1)}).$$

R. A. Rankin (Glasgow).

Djerasimović, B. Beitrag zur Untersuchung der Perron'schen Modularfunktion $M(\gamma)$ einer Irrationalzahl. Bull. Soc. Math. Phys. Serbie 6 (1954), 86–92. (Serbo-Croatian. German summary)

Perron's function $M(\gamma)$ is defined as follows. Let γ be any irrational and let A_n/B_n be the nth convergent to the regular continued fraction of γ. The numbers λ_n defined by

$$\left|\gamma-\frac{A_n}{B_n}\right|=\frac{1}{\lambda_n B_n^2}\quad (n=0, 1, 2, \cdots)$$

have a limit superior as $n\to\infty$, which is denoted by $M(\gamma)$. Any other irrational γ' equivalent to γ in the sense that $\gamma'=(a\gamma+b)/(c\gamma+d)$ $(ad-bc=1)$ is such that $M(\gamma')=M(\gamma)$. The author is concerned with the limit points of $M(\gamma)$. He shows first that every integer greater than 2 is such a limit point, as are also infinitely many rational numbers. At such points ϱ the numbers γ for which $M(\gamma)=\varrho$ have the power of the continuum. The same is true of certain cases in which ϱ is a quadratic surd. Every integer greater than 3 is a limit point of such points ϱ.

D. H. Lehmer (Berkeley, Calif.).

ANALYSIS

Have

★ **Jeffreys, Harold; and Jeffreys, Bertha Swirles. Methods of mathematical physics. 3d ed.** Cambridge, at the University Press, 1956. xi+714 pp. $15.00.

The first edition [1946] was reviewed in MR 8, 447, the second [1950] in MR 12, 12. For this edition only minor improvements and correction of known errors and misprints have been made.

Cordes, H. O. An inequality of G. Borg. Amer. Math. Monthly 63 (1956), 27–29.

G. Borg [Ark. Mat. Astr. Fys. 31A (1944), no. 1; MR 8, 70] established an inequality of the form $\int_0^L |y''/y|dx\geqq 4/L$, under certain hypotheses concerning the function $y(x)$. This inequality plays an important role in the study of Lyapunov zones of the stability of the differential equation $y''+\phi(x)y=0$. The author gives a simplified proof under more elementary assumptions concerning $y(x)$.

R. Bellman (Monterey, Calif.).

Bellman, Richard. On an inequality concerning an indefinite form. Amer. Math. Monthly 63 (1956), 108–109.

The author considers the function

$$\varphi(x)=(x_1^p-x_2^p-\cdots-x_n^p)^{1/p}\quad (p\geqq 1),$$

for values of the x_i in the region R defined by

$$x_i\geqq 0;\ x_1>(x_2^p+x_3^p+\cdots+x_n^p)^{1/p}.$$

He establishes the following inequality for x and y both in R:

$$\varphi(x+y)\geqq\varphi(x)+\varphi(y).$$

B. W. Jones (Boulder, Colo.).

Surányi, János. On the solubility of systems of linear inequalities. Eötvös L. Tud.-Egy. Kiadv. Term.-Tud. Kar Évk. 1952-53, 19–25 (1954). (Hungarian)

Let $l_i(x)=a_{i1}x_1+a_{i2}x_2+\cdots+a_{in}x_n$. Consider the inequalities

$$(1)\qquad l_i(x)\geqq 0\quad (1\leqq i\leqq n).$$

$\xi=(\xi_1, \xi_2, \cdots, \xi_n)$ is said to be a solution of (1) if not all ξ_i are 0 and $l_i(\xi)\geqq 0$ $(1\leqq i\leqq n)$. The author gives various necessary and sufficient conditions for the solvability of (1). He gives a different proof of a result of Blumenthal [Pacific J. Math. 2 (1942), 523–530; MR 14, 541]. The author remarks that after finishing his paper he found out that one of his results was simultaneously obtained by S. N. Černikov [Uspehi Mat. Nauk (N.S.) 8 (1953), no. 2(54), 1–13; MR 15, 293].

P. Erdös (Haifa).

Schoenberg, I. J. A note on multiply positive sequences and the Descartes rule of signs. Rend. Circ. Mat. Palermo (2) **4** (1955), 123–131.

The real sequence $a_0, a_1, \cdots, a_n$ ($a_0>0, a_n>0$) is said to be k-positive if the infinite matrix

$$A=\begin{vmatrix} a_0 & a_1 & a_2\cdots a_n & 0 & 0\cdots 0\cdots \\ 0 & a_0 & a_1\cdots a_{n-1} & a_n & 0\cdots 0\cdots \\ 0 & 0 & a_0\cdots a_{n-2} & a_{n-1} & a_n\cdots 0\cdots \\ \vdots & \vdots & & & \vdots \end{vmatrix}$$

has no negative minor of order $\leqq k$. Let $Z(S)$ be the number of zeros of $P(x)=a_0+a_1x+\cdots+a_nx^n$ in the sector S. It is first proved that if $Z(|\arg x-\pi|\leqq\pi/(k+1))=n$ then $a_0, a_1, \cdots, a_n$ is k-positive. Let $\nu(a)$ be the number of changes of sign of $a_0, a_1, \cdots, a_n$. Using the previous result and the fundamental theorem on variation-diminishing transformations, it is shown that if $Z(|\arg x-\pi|\leqq\pi/(k+2))\geqq n-k$ then $\nu(a)\leqq k$; if in addition $Z(x>0)=k$, then $\nu(a)=k$. This last result is due to S. Lipka [Math. Naturwiss. Anz. Ungar Akad. Wiss. **60** (1941), 70–82; MR **9**, 348]. Further, $Z(|\arg x-\pi|<\pi/\nu(a))\leqq n-\nu(a)$ and the result is true for $|\arg x-\pi|\leqq\pi/\nu(a)$ except when $\nu(a)=1$. The paper concludes with a demonstration of Obreschkoff's generalization of Descartes' rule of signs $Z(|\arg x|<\pi/(n-\nu(a)))\leqq\nu(a)$.

I. I. Hirschman (Levallois).

Denjoy, Arnaud. Les fonctions quasi-analytiques. C. R. Acad. Sci. Paris **242** (1956), 581–586.

The author proves two theorems stated in an earlier note [same C.R. **173** (1921), 1329–1331]. One gives a sufficient condition for a class of functions to be quasi-analytic; the other shows that, for a function with $|f^{(n)}(x)|\leqq M_n$ and $f^{(n)}(0)=0$ for all n, the series $\sum M_n^{-1/n}$ can converge arbitrarily slowly. *R. P. Boas, Jr.*

Kuipers, L.; and Meulenbeld, B. Symmetric polynomials with non-negative coefficients. Proc. Amer. Math. Soc. **6** (1955), 88–93.

If $f_1(x), \cdots, f_n(x)$ are each polynomials in x of degree $n+p$, $p\geqq 0$, such that all determinants of nth order from the coefficient matrix are non-negative, then the ratio $\det f_i(x_j)/\det(x_j^{i-1})$ is a symmetric polynomial in $x_1, \cdots, x_n$ with non-negative coefficients.

Following special cases which serve as lemmas, it is further shown that if $f(x)$, $g(x)$ are power series with non-negative coefficients, convergent in $0\leqq x<a$, then $(-1)^n\det^{-1}(x_j^{i-1})$ multiplied by the determinant whose ith row is

$$[1\ x_i\ x_i^2\cdots x_i^{n-3}\ f(ux_i)\ g(vx_1\cdots x_{i-1}x_{i+1}\cdots x_n)]$$

is again a power series with non-negative coefficients, convergent in $0\leqq x_1, \cdots, x_n<a$, $0\leqq u$, $a^{n-1}v\leqq 1$.

Finally, if $F(x, y)=\sum_{i,j=0}^{n+p} a_{i,j}x_iy_j$, where $p\geqq 0$ and all nth order determinants of $(a_{i,j})$ are non-negative, then

$$\det F(x_i, y_j)/\det(x_i^{k-1})\cdot\det(x_j^{k-1})$$

is a polynomial with non-negative coefficients and symmetric in the x's and y's separately.

A. B. Novikoff (Baltimore, Md.).

Theory of Sets, Functions of Real Variables

★**Denjoy, Arnaud. L'énumération transfinie. III. Études complémentaires sur l'ordination.** Gauthier-Villars, Paris, 1954. pp. 615–771. 2400 francs. Have

★**Denjoy, Arnaud. L'énumération transfinie. IV. Notes sur les sujets controversés.** Gauthier-Villars, Paris, 1954. pp. 773–971. 3200 francs. Have

A conclusione del suo grande, ammirevole trattato dedicato ad un approfondimento filosofico del concetto del transfinito, col fine di giustificarne l'uso ogni giorno più affermantesi nelle ricerche sui fondamenti delle matematiche [v. Livres I et II, 1946, 1952; MR **8**, 254; **15**, 408], l'Autore dedica i due ultimi fascicoli ad argomenti complementari e a questioni di carattere prevalentemente critico.

Una prima parte (pp. 615–665) sviluppa una nuova teoria degli insiemi finiti. Vengono assunti come idee primitive i concetti di ordine e di corrispondenza, e definito un insieme finito E quando: a) E è ordinabile (cioè può essere indicata una legge secondo la quale, di due elementi, p, q di E può sempre stabilirsi quello che precede e quello che segue, intendendo verificate le solite proprietà formali [simmetria e transitività] per i due termini precedere e seguire); b) tutti gli ordinamenti di E sono simili fra loro (cioè suscettibili d'esser posti in corrispondenza biunivoca conservante entrambi gli ordinamenti) (p. 636). Sono dimostrati poi i teoremi fondamentali: α) in ogni suo ordinamento, l'insieme finito E possiede sia un elemento iniziale, sia uno finale; β) della stessa proprietà gode ogni sottoinsieme di E; γ) E non può essere simile (qualunque sia il suo ordinamento) a nessuno dei suoi sottoinsiemi (distinti da E); δ) un insieme ordinabile e che in tutti i suoi ordinamenti possiede un elemento iniziale, è finito (questa proprietà è anche sufficiente affinchè E sia finito nel senso dell'Autore, e può quindi essere assunta come definizione d'insieme finito); ecc. L'Autore discute ampiamente la definizione da lui introdotta, confrontadola con le numerose definizioni date da altri autori.

Si passa poi a studiare il problema seguente (pp. 667–678). Sia E un qualunque insieme ordinate e sia $C(E)+D(E)$ una sua qualunque decomposizione in due sezioni disgiunte (cioè prive di elementi comuni): $C(E)$ sezione iniziale, $D(E)$ sezione finale (cfr. fasc. I, pp. 20–23). Si trovano le condizioni necessarie e sufficienti affinchè l'inserzione d'un nuovo elemento a fra $C(E)$ e $D(E)$ abbia per risultato un nuovo insieme $H(=C(E)+(a)+D(E))$ simile ad E [La necessità di opportune condizioni restrittive era già stata riconosciuta da Z. Chajoth [Fund. Math. **16** (1930), 132–133], del cui teorema l'Autore riporta la dimostrazione.]

Vengono poi date alcune proprietà degli insiemi dotati di rango (ensembles rangés), già profondamente studiati nel I fasc. (pp. 116–138), allo scopo di chiarire più completamente le differenze con la teoria di G. Cantor, quanto al significato dei termini: rango, numero ordinale, tipo di buon ordinamento (pp. 679–687).

L'Autore si diffonde ampiamente su un'analisi del cosidetto problema di Souslin (pp. 687–721), al quale egli aveva già dedicato numerose note [v. C. R. Acad. Sci. Paris **236** (1953), 435–439, 558–559, 641, 981–983; MR **15**, 409] terminanti in forma autocritica e dubitativa. — Il problema s'enuncia: un continuo ordinale E (insieme ordinato ed ordinalmente chiuso [cfr. fasc. I, pp. 20–23], privo di coppie d'elementi consecutivi), tale che ogni famiglia di sezioni disgiunte di E sia numerabile, è simile

al continuo lineare? Non è qui possibile entrare nello svolgimento, molto intricato e dettagliato, della questione, che viene esaminata anche dal ben noto punto di vista di G. Kurepa [v. ibid. **236** (1953), 564–565, 655–657; MR **15**, 409]. Ci limitiamo solo ad osservare che l'Autore non riesce a risolvere il problema, se non ammettendo l'uno o l'altro di vari particolari postulati (pp. 770, 713, 716), la cui ammissione non apparisce a vero dire giustificata. La mancata risoluzione del problema di Souslin lascia aperte varie questioni importanti relative ai numeri transfiniti, a proposito delle quali l'Autore è del parere che „ogni sforzo per edificare un corpo di proprietà positive, comuni ai numeri di seconda classe, è destinato a fallire; le conclusioni che fanno cadere una tale impressa, appariscono immancabilmente" (p. 721).

Ad ampia e profonda analisi critica è sottoposto l'assioma della scelta, sia in relazione specifica con gli argomenti particolari trattati (pp. 721–764) [v. anche C. R. Acad. Sci. Paris **236** (1953), 345–348; MR **15**, **408**], sia in generale (pp. 781–852). In linea di principio, l'Autore dichiara apertamente d'ammettere l'assioma (pp. 778, 782), non trova tuttavia perfettamente valide le dimostrazioni (basate, com'è noto, su quell'assioma) del teorema di Zermelo: ogni insieme è ben ordinabile (p. 846). La ragione di quest'originale opinione viene additata nel fatto che nessuna di quelle dimostrazioni riesce, a suo parere, ad evitare la considerazione dell'intera serie di numeri transfiniti: la possibilità di una simile considerazione viene ripetutamente negata (pp. 801, 847, 923, 965). — Secondo l'Autore, sarebbe preferibile sostituire il teorema di Zermelo con un altro postulato, indipendente da quello della scelta; la sua affermazione, in proposito, apparisce tuttavia estremamente cauta („Nous ne répugnons pas à ce postulat: il ne saurait être contradictoire avec la définition d'aucun ensemble d'admettre qu'il est bien ordonnable", v. p. 846)! Un tale atteggiamento pone la grave questione dell'effettiva esistenza, o meno, d'un generico transfinito della seconda classe. L'Autore ritiene di poter risolvere affermativamente la questione, dando una legge in virtù della quale ogni transfinito α della seconda classe può essere perfettamente determinato senza ricorrere alla preventiva determinazione dei transfiniti $<\alpha$ (pp. 813–852). Cioè: la formazione della totalità di detti transfiniti non richiede il buon ordinamento della classe stessa. La legge è trovata in una formale generalizzazione del metodo di rappresentazione, mediante sviluppi neperiani, delle permutazioni ben ordinate della successione dei numeri naturali (v. fasc. I, pp. 153–167).

La maggior parte del fasc. IV è occupata da una nota dal titolo: „Le concevable, le définissable, l'existant parmi les nombres et les ensembles" (pp. 853–965). Viene qui analizzato e descritto, con gran cura di dettagli, il meccanismo secondo cui il pensiero matematico perviene successivamente a formare concetti via via più generali ed astratti: quelli del numero intero (pp. 865–867), o del numero razionale (pp. 868–870), o del numero irrazionale (pp. 870–872), come pure quelli delle singole totalità di tali numeri (pp. 875–880, 883–914), o degli insiemi di punti di uno spazio, o di funzioni, o di numeri transfiniti (pp. 914–923). Nello studio di ciò che può essere definito, l'Autore considera come difficoltà essenziale il noto paradosso di Richard, del quale viene fatta un'analisi ampia ed approfondita e vengon date diverse spiegazioni: dapprima una di carattere generale, poi altre conformi ai vari possibili concetti del definito (pp. 926–954). Secondo l'Autore, ciò che può essere pensato può anche essere definito, ma non viceversa; analogamente ciò che può essere definito esiste, ma non viceversa (p. 955): per es. la totalità dei numeri transfiniti della seconda classe esiste, secondo l'Autore, ma non può essere definita (p. 965). Le idee di Borel e Lusin (parzialmente condivise dall' Autore), su ciò che è effettivamente definible e su ciò che esiste, vengono ampiamente discusse (pp. 956–960). In ogni caso l'Autore non dà alla parola esiste un significato puramente simbolico, ma reale cioè potenzialmente inerente al meccanismo del pensiero umano.

L'impressione conclusiva che abbiamo riportato dallo studio di questo trattato, è che l'originalissima posizione filosofica dell'Autore è fortemente impregnata di psicologismo, perciò in fondo non lontana da quella di Borel, al quale pure l'Autore non risparmia le critiche, alcune singolarmente acute. Interessante e molto suggestiva la valutazione quantitativa dell'«inconcepibile», come misura delle difficoltà inerenti a determinati meccanismi mentali (pp. 874–876), in antitesi con la concezione propria di Borel [Les nombres inaccessibles, Gauthier-Villars, Paris, 1952; MR **13**, 424]. — La logica viene concepita come immanente, non trascendente alla matematica, le idee primitive come facoltà psichiche: ciò vale soprattutto per l'idea d'ordine („assise de la pensée" (p. 619)), posta a fondamento sia del concetto di numero (nel trattato i numeri cardinali sono pressochè ignorati), sia del concetto di insieme finito. A questo proposito, le obiezioni mosse (p. 620) alla nota definizione di A. Tarski [Fund. Math. **6** (1924), 45–95] si sembrano attuare il massimo della compenetrazione fra logica e psicologia: infatti le ricerche più recenti della psicologia della età evolutiva hanno dimostrato che l'attività ordinatrice del pensiero è assolutamente inerente alla formazione dei concetti matematici nella mente del fanciullo [cf. J. Piaget, L'enseignement des mathématiques, Delachaux-Niestlé, Neuchatel-Paris 1955, pp. 11–33]. — L'Autore ritiene che il numero $x=0$ opp. 1 secondochè un certo evento (puramente casuale) che accade all'istante t_0, ha esito positivo oppure negativo, è definito quando $t \geqq t_0$, non è definito (ma esiste?) quando $t < t_0$ (p. 930). — Tutto ciò è, ci sembra, anti-aristotelico e richiama posizioni filosofiche moderne (per es. di Stuart Mill): infatti, secondo Aristotele, le definizioni esigono sola comprensibilità (cioè enunciazione logicamente corretta), non effettiva esistenza delle cose definite [*«Οἱ μὲν οὖν ὅροι οὐκ εἰσὶν ὑποθέσεις (οὐδὲ γὰρ εἶναι ἢ μὴ λέγονται), ἀλλ' ἐν ταῖς προτάσεσιν αἱ ὑποθέσεις. Τοὺς δ' ὅρους μόνον συνίεσθαι δεῖ· τοῦτο δ' οὐχ ὑπόθεσις, εἰ μὴ καὶ τὸ ἀκούειν ὑπόθεσίν τις εἶναι φήσει»*. Anal. post. Lib. I, cap. X, 9 (76^b); cf. G. Peano, Period. Mat. (4) **1** (1921), 175–189, ove il parere d'Aristotele è decisamente condiviso per quanto riguarda la matematica]. *T. Viola*

Schwartz, Laurent. L'énumération transfinie et l'oeuvre de M. Denjoy. Bull. Sci. Math. (2) **79** (1955), 78–96.

Recensione, ampia e scrupolosamente obiettiva, del trattato di A. Denjoy, L'énumération transfinie [Livres I–IV, Gauthier-Villars, Paris, 1946, 1952, 1954; MR **8**, 254; **15**, 408; ed anche la recensione precedente]. *T. Viola* (Roma).

Maurer, I. On the notion of power. Gaz. Mat. Fiz. Ser. A. **7** (1955), 601–611. (Romanian)

Expository paper. *E. Grosswald* (Philadelphia, Pa.).

Sierpinski, W. Quelques résultats et problèmes concernant la congruence des ensembles de points. Matematiche, Catania **10** (1955), 71–79.

Expository lecture.

Denjoy, Arnaud. Les ensembles parfaits linéaires de la première sorte. C. R. Acad. Sci. Paris **241** (1955), 1185–1189.

La présente note touche aux questions posées par la nature des ensembles parfaits inclus dans l'ensemble des discontinuités d'une fonction mesurable douée d'une dérivée symétrique finie (ou d'un couple de dérivés symétriques extrêmes finis, se confondant sur une plénitude avec une dérivée normale), sujet abordé par l'auteur aux mêmes C.R. **241** (1955), 617–620, 829–832 [MR **17**, 353]. Th. I concerne la dispersion de la somme de deux ensembles parfaits de la première sorte portés par un même axe, Th. II l'indice bilatéral minimum sur les ensembles de dispersion donnée, Th. III la dispersion de l'ensemble des distances mutuelles des points de deux ensembles de dispersions connues. La note contient les démonstrations.
C. Y. Pauc (Nantes).

Denjoy, Arnaud. Un problème de Lebesgue. C. R. Acad. Sci. Paris **241** (1955), 1237–1240.

Etant donné, dans un espace cartésien un ensemble ouvert R de frontière F, sur F une fonction continue $f(N)$, il s'agit de définir dans R une fonction $g(M)$ qui, complétée par $f(N)$ sur F, soit continue sur $R+F$. L'auteur donne pour $g(M)$ des solutions très générales et analytiques. *C. Y. Pauc* (Nantes).

Froda, Al. Sur la distribution des discontinuités des fonctions réelles. Com. Acad. R. P. Romîne **5** (1955), 31–36. (Romanian. Russian and French summaries)

In a previous paper [C. R. Acad. Sci. Paris **227** (1948), 1200–1201; MR **10**, 438] the author introduced and studied the notion of oscillation order α of a function f at a, denoted by $\omega^\alpha(f, a)$, where f is defined in an open set of R_n. The author now continues this study. He calls the discontinuity of f at a "reducible of order α" if $\omega^\alpha(f, a)=0$, while $\omega^{\alpha'}(f, a)>0$ for all $\alpha'<\alpha$; and he calls a discontinuity at a "irreducible" if $\omega^\alpha(f, a)>0$ for every α (of Cantor's classes I and II). Then he obtains the result that, for any given $\alpha>1$, the set of all discontinuities of f which are reducible of order α is at most denumerable and that the set of irreducible discontinuities of f is of the first category.
A. Rosenthal (Lafayette, Ind.).

Marcus, S. La dérivée approximative qualitative. Com. Acad. R. P. Romîne **3** (1953), 361–364. (Romanian. Russian and French summaries)

The author defines for real functions of a real variable the four qualitative approximate derivative numbers by means of the qualitative approximate limits, in the sense of his previous papers [same Com. **3** (1953), 9–12, 117–120; MR **17**, 20] of the usual difference quotients. Several results paralleling known theorems for ordinary derivatives are stated; for example, if two continuous functions have the same derivative numbers everywhere in an interval, except possibly at a countable set of points, then the functions differ by a constant. *M. M. Day.*

Kucharzewski, M. Die Differenzierbarkeit der homogenen Funktionen und die geometrischen Eigenschaften der Indicatrix von Carathéodory. Ann. Polon. Math. **1** (1955), 222–252.

A positive-homogeneous function $f(x)$ (with x in R_n) has under certain conditions (e.g., if $f(x)$ is continuous) an indicatrix I in R_n defined as the sets of points for which $f(x)=1$. The author shows, after some discussion of contingents and tangency in R_n, that in general the existence of a derivative for $f(x)$ at a point is equivalent to the existence of an n-dimensional tangent plane (not containing the origin) to I at the corresponding point.
U. S. Haslam-Jones (Oxford).

Kucharzewski, M. Eine Verallgemeinerung der Eulerschen Gleichung für homogene Funktionen. Ann. Polon. Math. **1** (1955), 326–337.

Let $f(x)$ be a function of $x=(x^1, x^2, \cdots, x^n)$ which is positive-homogeneous of degree μ, but not necessarily continuous. $f_k(x_0)$ is said to be the directed derivative (Richtungsableitung) at x_0 in the direction of the unit vector k if $\{f(x_0+\tau k)-f(x_0)\}/\tau\to f_k(x_0)$ as τ tends to $+0$. The author proves that if k and l are unit vectors and κ, λ are scalars such that $x_0=\kappa k+\lambda l$, then the existence of $f_k(x_0)$ implies that of $f_l(x_0)$ and the equality

$$\mu f(x_0)=\kappa f_k(x_0)+\lambda f_l(x_0).$$

He deduces certain necessary and sufficient conditions for $f(x)$ to satisfy Euler's equation in its ordinary form.
U. S. Haslam-Jones (Oxford).

Plamennov, I. Ya. On differential properties of measurable functions. Dokl. Akad. Nauk SSSR (N.S.) **104** (1955), 819–820. (Russian)

The note raises two questions connected with asymptotic derivation of continuous functions of one variable and with Jarnik's notion of essential derivates, and answers them negatively. In one case the set of counterexamples is stated to be not empty, in the other to be of the second category. *L. C. Young* (Madison, Wis.).

Plamennov, I. Ya. Some sufficient conditions for existence of an asymptotic differential. Dokl. Akad. Nauk SSSR (N.S.) **105** (1955), 416–418. (Russian)

Let $F(P)$ be a measurable, almost everywhere finite, real function defined in a plane measurable set E and let $A(P, k)$ denote the set of the points Q of E at which $F(P)-F(Q)-k|P-Q|>0$ (or alternatively <0). The author's main assertion is that at almost every point P of E at which F has no asymptotic differential, the set $A(P, k)$ has, for each k, (i) upper density at least $\frac{1}{2}$ in each angle with vertex at P, (ii) upper density 1 in at least one of any pair of adjacent such angles of sum $\leq\pi$. *L. C. Young* (Madison, Wis.).

See also: Sodnomov, p. 570; Rieger, p. 575; Nikodým, p. 594; Alexiewicz and Orlicz, p. 611; Banerjee, p. 639; Motchane, p. 650.

Theory of Measure and Integration

Dowker, Yael Naim. Sur les applications mesurables. C. R. Acad. Sci. Paris **242** (1956), 329–331.

In the notation of an earlier paper [Ann. of Math. (2) **62** (1955), 504–516; MR **17**, 353], and under the same assumptions, it is shown that if $\limsup m_n(A)>0$ whenever $m(A)>0$ then also $\liminf m_n(A)>0$ whenever $m(A)>0$. In conjunction with a theorem of Calderón [same C. R. **240** (1955), 1960–1962; MR **16**, 1008] it follows that there exists a finite invariant measure $\mu\sim m$ if and only if $\limsup m_n(A)>0$ whenever $m(A)>0$. (Calderón proved this theorem with lim inf in place of lim sup.) This completes the proof of the result (5) stated above. The sharpened version of (4) then follows from Lemma 9 of that paper. *J. C. Oxtoby.*

Nikodým, Otton Martin. Sur l'extension d'une mesure non archimédienne, simplement additive sur une tribu de Boole simplement additive, à une autre tribu, plus étendue. III. Extension de mesure. C. R. Acad. Sci. Paris **241** (1955), 1695–1696.

The notations are as in the review of the preceding notes [same C.R. **241** (1955), 1439–1440, 1544–1545; MR **17**, 468]. After assuming a certain completeness property of ϕ, the extendibility of μ to a finitely additive function on B' with values in ϕ is established by applying Banach's reasoning in the proof of the Hahn-Banach theorem.

H. M. Schaerf (St. Louis, Mo.).

Nikodým, Otton M. A theorem on infinite sequences of finitely additive real valued measures. Rend. Sem. Mat. Univ. Padova **24** (1955), 265–286.

The author's principal purpose is to establish the following theorem, proved for the case $p=1$ by E. Helly [Akad. Wiss. Wien. S.-B. IIa. **121** (1912), 265–297]. Let (B_p) denote the ring of sets in euclidean p-space ($p\geq 1$) generated by the set of products of p subintervals of [0, 1] each of which is half-open on the left. If, for $n=1, 2, 3, \cdots$ and $F \in (B_p)$, (i) $\mu_n(F)$ is a finitely additive, real-valued measure on (B_p), (ii) $|\mu_n(F)|\leq K$ a constant independent of n and F, then there exists an increasing sequence $k(n)$ of indices such that $\mu(F)=\lim \mu_{k(n)}(F)$ exists and the function $\mu(F)$ is a real-valued, finitely additive measure on (B_p) with $|\mu(F)|\leq K$. The extension to product spaces of countably infinite dimension is also indicated.

Making use of the correspondence between measures of the above type and functions of bounded Vitali variation, the author also gives the corresponding theorem on sequences of functions of uniformly bounded Vitali variation (the form of the original Helly theorem). The inductive proof follows the lines of the proof of the Helly theorem given by Wintner [Spektraltheorie der unendlichen Matrizen, Hirzel, Leipzig, 1929] but requires a number of preparatory lemmas. *W. R. Transue.*

Hlawka, Edmund. Zur formalen Theorie der Gleichverteilung in kompakten Gruppen. Rend. Circ. Mat Palermo (2) **4** (1955), 33–47.

This paper unifies and develops certain aspects of the theory of uniform distribution on a compact group G, a theory first studied by Eckmann.

Let $A=(a_{ik})$ be an infinite matrix of the type used in theory of summability, i.e., $\lim_{i\to\infty} \sum_k a_{ik}=1$, $\lim_{i\to\infty} a_{ik}=0$. A sequence $\{x_i\}$ of elements of G is said to be A-uniformly distributed (A-gleichverteilt) if, for every continuous real-valued function $\varphi(z)$,

$$\lim_{i\to\infty} \sum_k a_{ik}\varphi(x_k)=\int\varphi(x)dx.$$

Here the integral is the invariant integral on G, normalized so that the measure of the whole group is unity. When A is the $(C-1)$ matrix ($a_{ik}=1/i$ if $i\geq k$, $a_{ik}=0$ otherwise), the A is dropped and the sequence is simply said to be uniformly distributed. (This last is the case considered by Eckmann, and in the classical theory of uniform distribution, where G is a torus group.) The sequence is said to be A-g.g. distributed (gleichmässig gleichverteilt), if, uniformly for integers $h\geq 0$,

$$\sum_k a_{ik}\varphi(x_{k+h})\to\int\varphi(x)dx \text{ as } i\to\infty.$$

In addition to establishing certain basic results, some of which had been proved previously in a less general form, the author proves, among others, the following theorems. Theorem 4. Let $\{x_i\}$, $\{y_i\}$, be two sequences such that $\lim y_{i+1}^{-1}x_{i+1}x_i^{-1}y_i=e$, the identity. Then, if $\{x_i\}$ is g.g. distributed, so is $\{y_i\}$. Theorem 6. Let x_i be a sequence of elements of G. If each of the sequences $f^{(h)}=\{x_i^{-1}x_{i+h}\}$ is uniformly (g.g.) distributed, then every sequence $\{x_{ki+l}\}$ is uniformly (g.g.) distributed, where k, l are arbitrary natural numbers.

In the last part of the paper two independent proofs, one using ergodic theory, are given of the fact that almost all sequences are uniformly distributed. It is also shown that every dense sequence can be rearranged to form a uniformly distributed sequence, and that, if $\alpha>0$, almost all sequences are uniformly distributed with respect to the Cesaro matrix $(C-\alpha)$.

There is a misprint on p. 39, 1.16: "$D_f(x)$" should be "$D_f(a)$". *A. M. Macbeath* (Dundee).

Mikusiński, J. Une définition de distribution. Bull. Acad. Polon. Sci. Cl. III. **3** (1955), 589–591.

Distributions of finite order are here defined as classes of equivalent fundamental sequences of functions continuous on $-\infty<x<\infty$: $f_n(x)$ is fundamental if there exists an integer k and functions $F_n(x)$ such that (a) $f_n(x)=F_n^{(k)}(x)$ (kth derivative in the ordinary sense) and (b) $F_n(x)$ converge uniformly on each finite closed interval; sequences f_n, g_n are equivalent (and define the same distribution) if for some k, $f_n=F_n^{(k)}$, $g_n=G_n^{(k)}$ and F_n, G_n converge uniformly on each finite closed interval to the same limit. The author is unaware that this definition has been formulated independently by J. Korevaar [see the following review and papers cited there].

I. Halperin (Kingston, Ont.).

Korevaar, Jacob. Distributions defined by fundamental sequences. V. Integral of a product. Fourier series. Connection with Schwartz' theory. Nederl. Akad. Wetensch. Proc. Ser. A. **58**=Indag. Math. **17** (1955), 663–674.

This article concludes an interesting and readable exposition [for parts I–IV see same Proc. **58** (1955), 368–378, 379–389, 483–493, 494–503; MR **17**, 63, 354] of distributions, defined as limits of sequences of functions. Equivalence is shown between this definition and the Schwartz definition (linear functionals on a space of testing functions). The proof of equivalence, and most of the theory of the present exposition, are close to material contained in the books of Schwartz but the definition of definite integral $\int_a^b T dx$ (for some distributions T) appears to be new [cf. the following review]. Here $\int_a^b Tdx=G(b)-G(a)$ whenever (i) $T=G'$ and (ii) G coincides with a continuous point function locally at a and locally at b. Integration by parts, and Hadamard finite parts, are discussed for definite integrals of distributions.

Fourier series are also discussed for all distributions on $(0, 2\pi)$, using definite integrals of distributions.

I. Halperin (Kingston, Ont.).

Łojasiewicz, S.; Wloka, J.; und Zieleźny, Z. Über eine Definition des Wertes einer Distribution. Bull. Acad. Polon. Sci. Cl. III. **3** (1955), 479–481.

The following results are announced (proofs to appear in Rozprawy Matematyczne). Let $f=f(x)$ denote a distribution. I. The distribution-limit: $\lim_{\epsilon\to 0} f(\epsilon x+x_0)$, if it exists, is a constant function (coincides with $f(x_0)$ if f is, in a neighborhood of x_0, a point function continuous at

x_0). This limit is called the value of f at x_0, denoted $f(x_0)$ (definition attributed to S. Łojasiewicz). $\int_a^b f(x)dx$ is defined to be the value at 0 (if this exists) of distribution $(F(x+b)-F(x+a))$ with $F'=f$. II, If $F'=f$ and f has a value at x_0, then so does F. III. If f has values at a and b, then $\int_a^b f(x)dx$ is defined. IV. $\int_0^{2\pi} f(x)dx$ is defined if $f(x+2\pi)\equiv f(x)$; and $f(x)=\sum_{-\infty}^{+\infty} c_n e^{inx}$ with $c_n=(2\pi)^{-1}\int_0^{2\pi} f(x)e^{-inx}dx$ [cf. the preceding review]. *I. Halperin.*

Zieleźny, Z. Sur la définition de Łojasiewicz de la valeur d'une distribution dans un point. Bull. Acad. Polon. Sci. Cl. III. 3 (1955), 519–520.

On the real line $-\infty<x<\infty$ let $f\equiv f(x)$ denote a locally summable function, F a distribution and $\varphi(x)$ a continuous function possessing derivatives of all orders and vanishing outside some finite interval. For fixed a and $\alpha\neq 0$ define $f^{\alpha,a}$, $F^{\alpha,a}$ and $\varphi_{\alpha,a}$ as follows:

$$f^{\alpha,a}(x)=f(\alpha x+a),\ \varphi_{\alpha,a}(x)=\frac{1}{\alpha}\varphi\left(\frac{x-a}{\alpha}\right) \text{ and } F^{\alpha,a}(\varphi)=F(\varphi_{\alpha,a})$$

for all φ. If F coincides with a function f then $F^{\alpha,a}$ will coincide with $f^{\alpha,a}$. Always

$$(F^{\alpha,a})'=\alpha(F')^{\alpha,a},\quad (F^{(k)})^{\alpha,a}=\frac{1}{\alpha^k}(F^{\alpha,a})^{(k)};$$

$$\text{if } F=f^{(k)}, \text{ then } F^{\alpha,a}=\alpha^{-k}(f^{\alpha,a})^{(k)}.$$

Theorem (attributed to S. Łojasiewicz, see the 2nd preceding review). If $\lim_{\alpha\to 0} F^{\alpha,a}$ exists as a distribution T then T is a constant function (if $F=f$ and $f(x)$ is continuous at a, then T does exist and its constant value is $f(a)$).

Proof. $T^{\lambda,0}=T$ for all $\lambda\neq 0$ follows easily from the definition of T. Hence T is of finite order, $T=g^{(k)}$ for some continuous function $g(x)$ and some integer k and

$$(*)\qquad \lambda^k g(x)-g(\lambda x)=a_0(\lambda)+a_1(\lambda)x+\cdots+a_{k-1}(\lambda)x^{k-1}$$

a polynomial of degree $\leqq k-1$.The author succeeds in showing that $a_j(\lambda)$ is of the form $c_j(\lambda^j-\lambda^k)$ and hence that $g(x)$ is a polynomial of degree $\leqq k$. This implies that $T=g^{(k)}$=constant function.

I. Halperin (Kingston, Ont.).

Taylor, S. J. The α-dimensional measure of the graph and set of zeros of a Brownian path. Proc. Cambridge Philos. Soc. 51 (1955), 265–274.

The author proves that for almost every one-dimensional Brownian motion the set of zeros has Hausdorff dimension $\frac{1}{2}$ and zero $\frac{1}{2}$-dimensional measure. An auxiliary result is that almost every plane graph of such a motion has Hausdorff dimension $\frac{3}{2}$ and zero $\frac{3}{2}$-dimensional measure. *L. H. Loomis* (Cambridge, Mass.).

Davies, Roy O. A property of Hausdorff measure. Proc. Cambridge Philos. Soc. 52 (1956), 30–34.

If $\{E_n\}$ is an increasing sequence of sets, and $E=\sum E_n$, then $\Lambda_\delta^s E_n\to\Lambda_\delta^s E$. This requires that in defining Λ_δ^s we admit coverings by closed sets of diameter $\leqq\delta$.

H. D. Ursell (Leeds).

Besicovitch, A. S. On the definition of tangents to sets of infinite linear measure. Proc. Cambridge Philos. Soc. 52 (1956), 20–29.

E is said to be regular (irregular, have tangents $l(x)$ for $x\in E_1\subset E$) if every measurable subset E' of finite measure is regular (irregular, has tangent $l(x)$ for almost all $x\in E'E_1$). Most results for finite measure then extend to σ-finite measure: in particular, a regular set of σ-finite measure has tangents. On a simple arc of σ-finite measure the tangent exists on a subset T of infinite measure: the residual set N may be of infinite measure, but projects from almost all directions into a null-set.

Among sets of more-than-σ-finite measure there are a simple arc with no regular subset, and also a regular set having tangents. If $\psi(x)\prec\varphi(x)\preceq x$ as $x\to 0$, E is closed and of ψ-measure >0, then there exists $E'\subset E$, irregular and of φ-measure >0. If E is closed and of more-than-σ-finite linear measure, then there exists $\psi(x)\prec x$ such that E has ψ-measure >0. From these two results and an unpublished theorem of Davies it follows that an analytic (Suslin) set of more-than-σ-finite measure cannot be regular.

H. D. Ursell (Leeds).

Mickle, Earl J. Lebesgue area and Hausdorff measure. Rend. Circ. Mat. Palermo (2) 4 (1955), 205–218.

Let $T: x=x(w)$, $x=(x_1, x_2, x_3)$, $w=(u, v)\in Q=[0, 1, 0, 1]$, be any continuous mapping (path surface σ) from Q into the x-space S_3 and let $T(Q)$ denote the subset of S_3 covered by σ, and $A(T)$ the Lebesgue area of σ. The present paper gives a remarkably simple affirmative answer to the difficult question of defining a non-negative integer-valued multiplicity function $K(x, T, Q)$, $x\in S_3$ [$K\geqq 0$ in $T(Q)$; $K=0$ in $S_3-T(Q)$], such that the area $A(T)$ is equal to the integral of K in S_3 with respect to the 2-dimensional Hausdorff measure H^2; i.e.,

$$(*)\qquad A(T)=\iint_{S_3}(x, T, Q)dH^2.$$

[For a previous discussion see H. Federer, Bull. Amer. Math. Soc. 58 (1952), 306–378; MR 14, 149.]

Let Γ denote the collection of all H^2-measurable sets in S_3, let Γ_1 [Γ_2, Γ_3] denote the collection of all those sets $E\in\Gamma$ whose projection on the coordinate plane $\alpha_1=x_2x_3$ [$\alpha_2=x_3x_1$, $\alpha_3=x_1x_2$] have planar Lebesgue measure zero, and let $\Gamma_4=\Gamma_1\cap\Gamma_2$, $\Gamma_5=\Gamma_1\cap\Gamma_3$, $\Gamma_6=\Gamma_2\cap\Gamma_3$, $\Gamma_7=\Gamma_1\cap\Gamma_2\cap\Gamma_3$. Finally, for every $E\in\Gamma$ let $H_i(E)=\inf H^2(E-E_i)$ for all $E_i\subset\Gamma_i$ ($i=1, 2, \cdots, 7$). Then $H_i(E)$ ($i=1, 2, \cdots, 7$), is a measure function defined for all $E\in\Gamma$. Let π_i denote the projection of S_3 on α_i, $T_i=\pi_i T$, $E_i\subset Q$ the union of all Radó's essential maximal model continua (e.m.m.c.) for the plan mapping T_i, $i=1, 2, 3$. Let $e_1=E_1-(E_2\cup E_3)$, $e_2=E_2-(E_3\cup E_1)$, $e_3=E_3-(E_1\cup E_2)$, $e_4=(E_1\cap E_2)-E_3$, $e_5=(E_1\cap E_3)-E_2$, $e_6=(E_2\cap E_3)-E_1$, $e_7=E_1\cap E_2\cap E_3$. Then the collection $\{\gamma\}$ is defined of all maximal model continua γ for T in Q for which the following properties hold for at least one $i=1, 2, \cdots, 7$: (a) $\gamma\subset e_i$; (b) $T(\gamma)$ is a point of positive upper density in S_3 for the set $T(e_i)$ with respect to the measure H_i. The multiplicity function $K(x, T, Q)$ is then the number (possibly $+\infty$) of the components $T^{-1}(x)$ which are in $\{\gamma\}$. The multiplicity function K is invariant with respect to Fréchet equivalence and has property (*). This property is first proved for AC mappings having a Jacobian vector almost everywhere and then extended to all mappings T by virtue of a representation theorem of the reviewer. *L. Cesari* (Lafayette, Ind.).

Radó, Tibor. On multiplicity functions associated with Lebesgue area. Rend. Circ. Mat. Palermo (2) 4 (1955), 219–236.

Let $T: x=x(w)$, $x=(x_1, x_2, x_3)$, $w=(u, v)\in Q=[0, 1, 0, 1]$ be any continuous mapping (path surface σ) from Q into the x-space S_3 and let $T(Q)$ denote the subset of S_3 covered by σ, and $A(T)$ the Lebesgue area of σ. In the paper reviewed above E. J. Mickle defines a multiplicity

function $K(x, T, Q)$, $x \in E_3$, such that (*) $A(T) = \iint_{S_3} K(x, T, Q)dH^2$, where H^2 denotes the 2-dim. Hausdorff measure in S_3. Mickle's multiplicity function $K(x, T, Q)$ is defined by means of seven measure functions H_i ($i=1, 2, \cdots, 7$), derived from H^2. In the present paper it is shown that by using the first three of the functions H_i it is possible to define a class of multiplicity functions analogous to K for which all (*) holds. For every point P of the unit sphere U in S_3 let $S_2(P)$ be the plane through (0, 0, 0) which is perpendicular to the vector OP, let π_P be the projection of S_3 on $S_2(P)$, E_P the collection of all e.m.m.s. relative to the plane mapping $\pi_P T$, Γ_P the class of all sets $E \in \Gamma$, $E \subset S_3$, whose projection on $S_2(P)$ has planar Lebesgue measure zero. Finally, for $E \subset S_3$ let $H_P^2(E) = \inf H^2(E-E_0)$ for all $E_0 \in \Gamma_P$, and $D_P(E)$ the set of all points $x \in E$ having an upper 2-dim. density positive with respect to the measure function $H_P^2(E)$. For any set $\sigma \subset U$ let $K_\sigma(x, T, Q)$ denote the number (possibly $+\infty$) of all maximal continua of constancy γ of T in Q with $\gamma \subset E_P$, $T(\gamma) \in D_P T E_P$ for at least a point $P \in \sigma$. There arises the problem of characterizing those sets $\sigma \subset U$ for which

$$(**) \qquad A(T) = \iint_{S_3} K_\sigma(x, T, Q)dH^2.$$

The author shows that (**) holds provided σ contains the points (1, 0, 0), (0, 1, 0), (0, 0, 1) [for instance σ could be the union of these points]. Also (**) holds if σ is countable and its closure contains the three points above. A theorem of the reviewer's on surfaces with $A(T) < +\infty$ is used in the proofs. *L. Cesari* (Lafayette, Ind.).

De Giorgi, Ennio. Nuovi teoremi relativi alle misure $(r-1)$-dimensionali in uno spazio ad r dimensioni. Ricerche Mat. **4** (1955), 95–113.

In a previous note [Ann. Mat. Pura Appl. (4) **36** (1954), 191–213; MR **15**, 945] the author established in Euclidean r-space, for a set E possessing a finite $(r-1)$-dimensional perimeter in his sense, a formula of the type $\int_E \operatorname{grad} g(x)dx = \int g(x)d\Phi$, where Φ is vector-valued. He now defines a reduced boundary of E and shows that on it the total variation of Φ is any of the usual $(r-1)$-dimensional measures, while outside it this total variation is zero. *L. C. Young* (Madison, Wis.).

Have

★ **Cesari, Lamberto. Surface area.** Annals of Mathematics Studies, no. 35. Princeton University Press, Princeton, N.J., 1956. x+595 pp. $8.50.

Cesari's "Surface area" appears just ten years after the completion of Radó's "Length and area" [Amer. Math. Soc. Colloq. Publ., v. 30, New York, 1948; MR **9**, 505].

Since the latter has inspired many researches during the intervening years, it has rendered Cesari's task all the more arduous, by the wealth of new material to be presented as well by the high standard of its proved excellence. This wealth of material is evidenced by Cesari's enormous bibliography, which occupies eighteen closely typed pages and which lists six or seven hundred references to papers, books and to mimeographed lectures or reports, by some two hundred authors; and often a single reference is used for a whole series of papers. Needless to say, a sizeable proportion of the papers, over 120, are due, either to Cesari himself, or to the school of Tibor Radó which so largely inspired him. Nevertheless, Cesari's book is only slightly longer than Radó's, and only so because of the bibliography and two short appendices. In fact, "Surface area" is not planned as an Encyclopedia, but rather as a self-contained account of a successful campaign, in which from beginning to end, and indeed at every stage, the most direct strategic route has been followed. In this manner some topics have had to be bi-passed, but with good reason, for instance the interesting work of J. W. T. Youngs on Fréchet equivalence, in which there are many pitfalls.

"Surface area" centres around three theorems; these are analogous to classical theorems about arc length, and the author has aimed at freeing them from difficulties with which all previous expositions, his own included, have been confronted. One such difficulty was that one of the main tools previously used, namely Morrey's theorem, requires the parameter domain to be a 2-cell or 2-sphere; another was that the proofs leaned heavily on Fréchet equivalence, a notion which plays no part in two of the statements. The reader is not conscious of the author's arduous task, for it is carried out skillfully, with apparent handicaps being turned into points of intrinsic interest.

Now for the first time, the greater part of the theory is obtained without recourse either to Morrey's theorem or to Fréchet equivalence, and the parameter domain is permitted to be any admissible set A, i.e. any plane set expressible as a relatively open subset of a finite sum of disjoint closed Jordan regions. The symbol (T, A) denotes a continuous mapping T of an admissble set A into Euclidean 3-space, and the first eight chapters deal with such continuous mappings without any reference to Fréchet equivalence. The whole book consists of ten chapters with two appendices and a bibliography. The study of the three theorems includes, among other things, the Cavalieri inequality and the identity of Lebesgue, Geöcze and Peano areas; the latter, and also two of the theorems, appear for the first time in book form. In various chapters, there are additional starred sections, in which cognate material is presented, or else alternative treatments are suggested, or referred to in Radó's book or elsewhere.

Chapter I is an introduction and summary of results. Chapter II defines the Lebesgue area L in terms of approximation by piecewise linear mappings and semi-continuity, in conjunction with a suitable specification of the convergence of (T_n, A_n); L is shown to be subject to the Kolmogoroff and projection principles and to the properties of semi-continuity and additivity; and reference is made to alternative definitions and to the Geöcze problem. Chapter III concerns a rival notion of area, in which the reviewer has a family interest; it is independent of convergence of (T_n, A_n) to (T, A) and is based on the plane projections of (T, A) and on the topological index; its various forms include the Geöcze areas U and V, the Peano area P. There follow two starred sections about semi-continuous collections and about properties of the plane.

Chapter IV deals with plane mappings and notions of bounded variation and absolute continuity, and naturally also with the notion of a multiplicity function, which goes back, in its crudest form, to Banach, and which, suitably refined at the hands of Radó and others, has been one of the main tools of area theory. The chapter also contains two starred sections, on local properties of plane mappings (a study of elements on each of which T is constant) and on a characterization of absolutely continuous plane mappings; the former introduces concepts due to Radó. Chapter V is devoted to "the first theorem," which includes the assertion, first established by Cesari himself, that $L(T, A)$ is finite if and only if the projections of T on the coordinate planes are of bounded variation

in A; the original proof was too late for inclusion in Radò's book, so that the details now appear for the first time in book form; they are based on the so-called "4-lines theorem;" an alternative argument, applicable to mappings into higher-dimensional spaces, is given in Appendix A at the end of the book; a proof valid for surfaces in higher space has also been published by Federer.

Chapter VI develops notions leading up to Cavalieri's inequality, an inequality which has been associated with Eilenberg and Szpilrajn-Marczewski but which essentially occurs in elementary Analysis; it constitutes a basic tool in area theory today, for instance in connection with work of Aronszajn and Choquet which is frequently quoted but still unpublished; the notions needed to lead up to it are parts of the theory of prime ends, and a definition of length, strongly connected with the reviewer's work. Chapter VII is mainly devoted to deriving, in a number of stages, the deep identity $L=U=V=P$, first proved by Cesari and Cecconi, which constitutes the heart of the book; a final section of the chapter discusses the Lebesgue area as a measure and presents a new result of Rosenthal.

Chapter VIII introduces generalized Jacobians and corresponding transformation formulae, and establishes, mainly as a result of the identity $L=V$, "the second theorem" which connects generalized Jacobians with L; this is similar to developments in Radó's book except for the greater generality of A. Chapter IX disposes briefly of the notion of Fréchet equivalence and proves Morrey's theorem with the help of Cesari's equi-continuity criteria and the reviewer's ε, δ gratings. Chapter X is devoted to "the third theorem," which is due to Cesari and which asserts that, for any (T, A) of finite Lebesgue area, there is a Fréchet equivalent generalized conformal mapping with nice properties (the terminology is here partly in conflict with that of Radó's book). Finally, Appendix B gives an account of the Weierstrass integral over a surface of finite area, which can be used to derive an isoperimetric theorem of Radó and J. W. T. Youngs and to extend the formulae of Gauss and Stokes and the well known semi-continuity theorem of McShane.

There is one respect in which Cesari's "Surface area" most resembles Radó's "Length and area," namely in that it constitutes, like the latter, a new departure in area theory, likely to have a profound influence on future developments. Indeed the fact that the subject is so well rounded off must not deceive the reader into imagining that its study is now complete. For the purposes of Analysis, it was found necessary to add to the analogous three theorems about arc length, a deep theorem, due to Marston Morse, on the uniform parametrization of a set of curves, which so far has not been transposed to the case of surfaces. Moreover, what is perhaps most interesting in Cesari's account—the substitution of an arbitrary admissible set A for the parameter domain and the partial avoidance of Fréchet equivalence—may be only a first step towards including, for instance, surfaces of infinite topological types, needed for the complete solution of Plateau's problem.

L. C. Young (Madison, Wis.).

See also: Nikodým, p. 594; Edwards, p. 645; Ionescu Tulcea, p. 645.

Functions of a Complex Variable, Generalizations

★ **Gončarov, V. L. Teoriya funkcii kompleksnogo peremennogo. [Theory of functions of a complex variable.]** Gosudarstv. Učebno-Pedagog. Izdat., Moscow, 1955. 351 pp. 3.25 rubles.

This textbook for teachers' colleges corresponds roughly in content to Churchill's "Introduction to complex variables and applications" [McGraw-Hill, New York, 1948; MR **10**, 439], except that it contains almost no applications. *R. P. Boas, Jr.* (Evanston, Ill.).

Parodi, Maurice. Compléments à un théorème de Pellet. Bull. Sci. Math. (2) **79** (1955), 101–105.

The theorem of Pellet states that the real polynomial

$$f(z)=z^n+a_1z^{n-1}+\cdots+a_pz^{n-p}+\cdots+a_{n-1}z+a_n\ (a_n\neq 0),$$

has $n-p$ zeros inside the unit circle $|z|<1$ and p zeros outside the unit circle if $|a_p|>1+|a_1|+\cdots+|a_{p-1}|+|a_{p+1}|+\cdots+|a_n|$. The zeros of $f(z)$ are the characteristic values of the matrix $A=\|c_{jk}\|$, where $c_{jk}=0$ except that $c_{j,j+1}=1$ $(j=1,\cdots,n-1)$ and $c_{nk}=-a_{n-k+1}$ for $k=1,\cdots,n$. From some theorems on such characteristic values, one concludes that the zeros of $f(z)$ outside the unit circle lie in the region common to the circles

$$|a_1+z|\leq|a_2|+\cdots+|a_n|,\quad |z|\leq 1+|a_p|,$$

if $p\neq 1$ and $p\neq n$. The second circle is omitted if $p=1$ and replaced by the circle $|z|\leq|a_n|$ if $p=n$. By an inversion in the unit circle, the corresponding results are obtained for the zeros inside the unit circle. Sharper results are obtained for the lacunary polynomial $f(z)=z^n+a_pz^{n-p}+\cdots+a_{n-1}z+a_n$. *M. Marden* (Milwaukee, Wis.).

Mahler, K. On the Taylor coefficients of rational functions. Proc. Cambridge Philos. Soc. **52** (1956), 39–48.

The author proves: If $F(z)=\sum_{h=0}^{\infty} f_hz^h$ is a rational function of z and infinitely many f_h vanish, then there exist integers L, L_1 $(0\leq L_1<L)$ such that $f_h=0$ for all sufficiently large $h\equiv L_1 \pmod L$. This theorem is contained in a theorem of C. Lech [Ark. Mat. **2** (1953), 417–421; MR **15**, 104] which discusses power series $\sum f_hz^h$ for which f_h satisfies a recurrence relation $f_h=\alpha_1f_{h-1}+\cdots+\alpha_nf_{h-n}$, $h=n, n+1, \cdots$. Lech's result is slightly stronger.

J. Lehner (Los Alamos, N.M.).

Helson, Henry. On a theorem of Szegö. Proc. Amer. Math. Soc. **6** (1955), 235–242.

The theorem of Szegö is the following: If the coefficients a_n in $f(z)=\sum_0^\infty a_nz^n$ are all taken from a finite set $(b_1,\cdots,b_m)$, and if $f(z)$ is continuable beyond the unit circle, then $f(z)$ is rational [S.-B. Preuss. Akad. Wiss. **1922**, 88–91]. (This is equivalent to having $\{a_n\}$ ultimately periodic.) Duffin and Schaeffer [Amer. J. Math. **67** (1945), 141–154; MR **6**, 148] show that the theorem is valid if $|f(z)|$ is bounded on some sector of $|z|=1$, rather than continuable. Helson [Proc. Amer. Math. Soc. **4** (1953), 686–691; MR **15**, 309] obtained an analogue for harmonic functions with bounded integral means. The present paper establishes the following, which includes all of these as special cases: Let $u(r,x)=\sum_{k=-\infty}^{\infty} a_kr^{|k|}e^{ikx}$ have only finitely many distinct coefficients and for some arc(α,β) of the circle satisfy

$$\int_\alpha^\beta |u(r,x)|dx\leq M<\infty\quad (r<1).$$

Then $\{a_k\}$ is ultimately periodic to the right and ultimately

periodic to the left. Helson's methods, which depend on the theory of measure and linear spaces, are different from most of the earlier papers. The argument uses the theorem of F. and M. Riesz on Fourier-Stieltjes coefficients, which Helson has recently proved anew [Colloq. Math. 3 (1955), 113–117; MR 16, 1016].

R. M. Redheffer (Los Angeles, Calif.).

Ricci, Giovanni. Sulle serie di potenze lacunari prolungabili e ultraconvergenti. Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) 18 (1955), 27–31.

Preliminary announcement of the results of the paper reviewed below.

Ricci, Giovanni. Prolungabilità e ultraconvergenza delle serie di potenze. Modulazione del margine delle lacune. Rend. Mat. e Appl. (5) 14 (1955), 602–632.

Let the series (1) $\sum a_n z^n$ have radius of convergence one. For $\theta>0$, a sequence of segments (p_h, q_h) on the positive real axis, with $p_h\to\infty$ and $q_h>(1+\theta)p_h$, constitutes a θ-sequence of H-O (Hadamard-Ostrowski) gaps for (1) provided the condition (2) $\lim\sup |a_n|^{1/n}<1$ is satisfied for the indices n that fall into the intervals (p_h, q_h). It is a θ-sequence of F-P (Fabry-Pólya) gaps provided the indices n that fall into the intervals (p_h, q_h) can be divided into two sets, both infinite, such that (2) holds for the first set but not for the second and such that the number ν_h of indices n in (p_h, q_h) which belong to the second set satisfies the condition $\nu_h=o(q_h-p_h)$. The author defines Λ, the order of H-O lacunarity of (1), as the supremum of the values θ for which (1) has a θ-sequence of H-O gaps (if no θ-sequence of H-O gaps exists, then $\Lambda=0$); similarly, he defines the order Λ^* of F-P lacunarity. He shows that if Λ and Λ^* are any two values in $[0, \infty]$, there exists a series (1) whose orders of lacunarity are Λ and Λ^*.

It is known that if (1) belongs to the class R of series which can be continued analytically beyond the unit disk, then $\Lambda^*=0$. From this the author deduces, for example, that if (1) is in R and has a finite, positive order Λ of H-O lacunarity, then (1) does not possess a Λ-sequence of H-O gaps. More generally: if (1) is in R and has a sequence $\{(p_h, q_h)\}$ of H-O gaps, then there exists a positive number d such that $\{((1-d)ph, (1+d)q_h)\}$ is also a sequence of H-O gaps for (1). Other theorems concern the lengths of the segments between consecutive H-O gaps of series (1) of class R. *G. Piranian* (Ann Arbor, Mich.).

Turán, Pál. Some remarks on theory of functions and theory of series. Eötvös L. Tud.-Egy. Kiadv. Term.-Tud. Kar Évk. 1952-53, 5–13 (1954). (Hungarian)

The author discusses various problems and results on the theory of functions and summability. Among others he proves the following theorem. Let α be a complex number with $\mathfrak{J}(\alpha)\neq 0$; then there exists an analytic function $f(z)=\sum_{n=0}^{\infty} a_n z^n$ whose only singular point is at $z=1$, and for which

$$(*)\qquad \sum_{n=0}^{\infty} \frac{f^{(n)}(\alpha)}{n!}(1-\alpha)^n$$

diverges (i.e. the Taylor expansion around α diverges at $1-\alpha$). The author shows that $f(z)=e^{1/(z-1)}$ has the required property. He further asks if such an $f(z)$ exists if $-1<\alpha<0$? He remarks that if $0<\alpha<1$ then by a theorem of Hardy and Littlewood (*) converges (here we only have to assume that $f(z)$ is analytic for $|z|<1$).

P. Erdős (Haifa).

Džrbašyan, M. M.; and Tamadyan, A. P. On best approximation by entire functions in a complex region. Dokl. Akad. Nauk SSSR (N.S.) 104 (1955), 345–348. (Russian)

The authors state numerous theorems, both direct and inverse, dealing with the best approximation A_σ by means of entire functions of given order ϱ of type not exceeding σ, to functions belonging to various classes defined in appropriate angular domains. For $\varrho=1$, some of their results reduce to known theorems about approximation on the real axis by entire functions of exponential type. The following statements may serve as samples. If $\varrho>1$, $f(z)$ is bounded and continuous, and $\sup|f'(z)|\leqq M$ in $|\arg z|<\frac{1}{2}\pi(1-1/\varrho)$ and $|\arg z+\pi|<\frac{1}{2}\pi(1-1/\varrho)$, then (for this domain) $A_\sigma\leqq CM\sigma^{1/\varrho}$, where C is independent of σ and f. If $\varrho\geqq\frac{1}{2}$, and $A_\sigma<Ce^{-\sigma\gamma}$ for every positive σ, with γ and C independent of σ, in $|\arg z|\geqq\frac{1}{2}\pi/\varrho$, then $f(z)$ is analytic and bounded in the region containing the negative real axis and bounded by the curve $r^\varrho\cos\varrho\theta=\gamma$.

R. P. Boas, Jr. (Evanston, Ill.).

Clunie, J. The maximum modulus of an integral function of an integral function. Quart. J. Math. Oxford Ser. (2) 6 (1955), 176–178.

Let g and h be entire functions and set $f=g(h)$. Let $h(0)=0$. Denote $\max|f(z)|$ on $|z|=r$ by $F(r)$, etc. Pólya [J. London Math. Soc. 1 (1926), 12–15] proved the existence of a number c, $0<c<1$, such that $F(r)\geqq G\{cH(\frac{1}{2}r)\}$. The author sharpens this result as follows. Let $N(r)$ be the index of the term in the power series of $h(z)$ which has maximum modulus on $|z|=r$. Then outside a set of finite logarithmic length one has for $r\geqq 1$

$$F\{r\exp(kN^{-3/2}\log^{5/2}N)\}\geqq G\{H(r)\},$$

where k is a constant depending on h. An inequality holding in the exceptional intervals is also given.

J. Korevaar (Madison, Wis.).

Janowski, W. Le maximum des coefficients A_2 et A_3 des fonctions univalentes bornées. Ann. Polon. Math. 2 (1955), 145–160 (1956).

For $M>1$, let F_M be the family of all univalent functions in $|z|<1$ satisfying $|F(z)|<M$ and $F(z)=z+A_2z^2+A_3z^3+\cdots$. The author gives another proof, using variational methods, of the theorems

$$\text{(I)}\qquad |A_2|\leqq 2(1-M^{-1})$$

and

$$\text{(II)}\qquad |A_3|\leqq\begin{cases}2\lambda^2+1-4\lambda M^{-1}+M^{-2} & \text{for } M\geqq e,\\ 1-M^{-2} & \text{for } M\leqq e,\end{cases}$$

where λ is the larger of the two roots of $\lambda\log\lambda=-M^{-1}$; these limits are attained for some functions in F_M. Theorem (I) was originally given by Pick [Akad. Wiss. Wien S.-B. IIa. 126 (1917), 247–263]. Theorem (II) was essentially proved by Löwner [Math. Ann. 89 (1923), 102–121], and later by Schaeffer and Spencer [Duke Math. J. 12 (1945), 107–125; MR 6, 206]. *G. Springer.*

Janowski, W. Le maximum des coefficients B_2 et B_3 des fonctions univalentes K-symétriques bornées. Ann. Polon. Math. 2 (1955), 161–169 (1956).

In this paper the author adds an assumption to the functions in F_M [see the preceding review] to form a family Φ_M. The additional hypothesis is that the functions F be K-symmetric (K a positive integer), and $F(z)=$

$z+B_2z^{K+1}+B_3z^{2K+1}+\cdots$. He then proves the theorems

$$\text{(I)}\qquad |B_2|\leqq 2K^{-1}(1-M^{-K})$$

and

$$\text{(II)}\quad |B_3|\leqq\begin{cases} K^{-1}(1-M^{-2K}) \text{ for } M\leqq \exp[2/(K+1)] \\ K^{-1}(2\lambda^2+1+M^{-2K}-4\lambda M^{-K}) \text{ for} \\ \qquad M\leqq \exp[2/(K+1)], \end{cases}$$

where λ is the larger of the two roots of the equation

$$\lambda\log\lambda+\lambda(K-1)/(K+1)=-M^{-K};$$

these limits are attained for functions in Φ_M.

G. Springer (Lawrence, Kan.).

Janowski, W. Le maximum de la partie imaginaire des fonctions univalentes bornées. Ann. Polon. Math. 2 (1955), 182–200 (1956).

For functions F in F_M [see the second preceding review] the author proves that the imaginary part of F satisfies $\Im\{F(r)\}\leqq\varrho_M\sin\varphi_M$, where ϱ_M $(0<\varrho_M<r)$ and φ_M $(\sin\varphi_M>0)$ are solutions of

$$\frac{[1+(\varrho/M)^2]\sin\varphi-2\varrho/M}{(2\varrho/M)\sin\varphi-[1+(\varrho/M)^2]}\log\left(\frac{1-\varrho/M}{1+\varrho/M}:\frac{1-r}{1+r}\right)+\log M\frac{1-r^2}{r(M/\varrho-\varrho/M)}=0,$$

$$\varphi=\left[\log^2\left(\frac{1-\varrho/M}{1+\varrho/M}:\frac{1-r}{1+r}\right)-\log^2\left(M\frac{1-r^2}{r(M/\varrho-\varrho/M)}\right)\right]^{\frac{1}{2}},$$

for which $\varrho\sin\varphi$ is largest. He also shows that this limit is attained for some function in F_M. For $F\in F_\infty$, $\Im\{F(r)\}\leqq\varrho_\infty\sin\varphi_\infty$, where ϱ_∞ and φ_∞ satisfy

$$\log\varrho_\infty=\log(r/(1-r^2))+\sin\varphi_\infty\log((1+r)/(1-r))$$
$$\varphi_\infty=\cos\varphi_\infty\log((1+r)/(1-r)),$$

and this limit is also attained for a function in F_∞. The proof is based upon variational methods. *G. Springer.*

Janowski, W. Sur les fonctions univalentes K-symétriques. Ann. Polon. Math. 2 (1955), 201–208 (1956).

The author obtains estimates similar to those given in the paper reviewed above for the imaginary part of functions in F_M (F_∞) which are K-symmetric.

G. Springer (Lawrence, Kan.).

Lebedev, N. A. Some estimates for functions regular and univalent in a circle. Vestnik Leningrad. Univ. 10 (1955), no. 11, 3–21. (Russian)

Suppose that $f(\zeta)=\alpha\zeta+\cdots$, $\alpha>0$, is regular univalent and $|f(\zeta)|<1$ in $|\zeta|<1$. Starting from the classical Loewner differential equation [Math. Ann. 89 (1923), 103–121], the author proves that for $|z|<1$,

$$\text{(1)}\qquad \left|\ln\frac{f(z)(1-|z|^2)}{\alpha z(1-|f(z)|^2)}\right|\leqq\ln\frac{(1+|z|)(-|f(z)|)}{(1-|z|)(1+|f(z)|)},$$

$$\text{(2)}\qquad \left|\ln\frac{zf'(z)}{f(z)}\right|\leqq\ln\frac{(1+|z|)(1-|f(z)|)}{(1-|z|)(1+|f(z)|)},$$

$$\text{(3)}\qquad \left|\ln\frac{\alpha z^2f'(z)}{f(z)^2}\right|\leqq\ln\frac{1-|f(z)|^2}{1-|z|^2},$$

$$\text{(4)}\qquad |\ln\alpha f'(z)|\leqq\ln\frac{|z|^2(1-|f(z)|^2)}{(1-|z|^2)|f(z)|^2}.$$

If z is fixed, (1) gives the region of values for the point $(\alpha, \ln f(z)/z)$. If z and $|f(z)|$ are fixed then (2) and (3) give the regions of values for the quantities on the left side. The inequality (4) gives only an estimate, but the author obtains the region of values for $\ln\alpha f'(z)$. In this latter case explicit equations are given for the boundary, but they are too complicated to reproduce here.

A large variety of corollaries are deduced from these four inequalities, but the author seems to be unaware that several of them were obtained earlier by R. M. Robinson [Trans. Amer. Math. Soc. 52 (1942), 426–449; MR 4, 77].

A. W. Goodman (Lexington, Ky.).

Ašnevic, I. Ya.; and Ulina, G. V. On regions of values of analytic functions represented by a Stieltjes integral. Vestnik Leningrad. Univ. 10 (1955), no. 11, 31–42. (Russian)

Let $E=E[G(\zeta, t), a, b]$ denote the class of functions $f(\zeta)$, regular in $|\zeta|<1$, with the Stieltjes representation

$$f(\zeta)=\int_a^b G(\zeta, t)d\mu(t),\qquad \int_a^b d\mu(t)=1,\qquad \mu(t)\uparrow,$$

where the kernel $G(\zeta, t)$ is a fixed function regular in ζ for $|\zeta|<1$ and continuous in t for $a\leqq t\leqq b$. Various choices of $G(\zeta, t)$ lead to classical normalized sets of functions, thus: (1) $(e^{it}+\zeta)/(e^{it}-\zeta)$ gives the set of functions with positive real part; (2) $\zeta/(1-2t\zeta+\zeta^2)$ gives the set of functions typically real and regular $|\zeta|<1$; (3) $-(1-\zeta^2)^2/\zeta(1-2\zeta t+\zeta^2)$ gives the set of functions typically real in $|\zeta|<1$ and regular except for a single pole at $\zeta=0$; and (4) $\log(1-e^{-it}\zeta)$ gives the set of functions $\frac{1}{2}\log(\zeta/f(\zeta))$, where $f(\zeta)$ is starlike and univalent in $|\zeta|<1$.

The authors prove that for fixed z and $G(\zeta, t)$, the domain of variability of $f(z)$, for $f(\zeta)\in E$ is the smallest convex cover of the curve $G(z, t)$, $a\leqq t\leqq b$. From this central theorem they easily deduce a variety of known results and a few new ones.

Some of their references are incorrect. The kernel (1) is due to Herglotz [Ber. Verh. Sächs. Ges. Wiss. Leipzig. Math.-Phys. Kl. 63 (1911), 501–511], and (2) is due to Robertson [Bull. Amer. Math. Soc. 41 (1935), 565–572]. Theorem 1 of § 4 of this paper was first proved by Rogosinski [Math. Z. 35 (1932), 93–121]. The reviewer cannot agree with the "only" assertion made in Theorem 1 of § 1 of the paper. *A. W. Goodman* (Lexington, Ky.).

Tammi, Olli. Note on symmetric schlicht domains of bounded boundary rotation. Ann. Acad. Sci. Fenn. Ser. A. I. no. 198 (1955), 10 pp.

Let $2\leqq k\leqq 4$ and let $f(z)=z+a_2z^2+a^3z^3+\cdots$ be a schlicht function, with real coefficients, mapping the unit circle onto a domain whose boundary curve has a tangent angle with total variation not exceeding $k\pi$. By means of a representation for $f(z)$ in terms of a Poisson-Stieltjes integral, it is shown that $|a_4|\leqq(k^3+8k)/24$, with equality holding only for certain maps onto infinite angular domains. *P. R. Garabedian* (Stanford, Calif.).

Eggleston, H. G. A property of bounded analytic functions. Comment. Math. Helv. 30 (1956), 139–143.

Let $f(z)$ be a bounded regular function in $|z|<1$. The set of all $e^{i\theta}$ with $0<\theta\leqq 2\pi$, for which $\lim_{r\to 1} f(re^{i\theta})$ exists is denoted by $F(f)$. The set of points ζ on $|z|=1$ with the property that $\lim f(z)$ exists as $z\to\zeta$, with $|z|<1$, is denoted by $G(f)$. Clearly, $G(f)\subseteq F(f)$. It is shown that the set $F(f)-G(f)$ is of first category. This result is applied to show that there exist bounded regular functions $f(z)$ in $|z|<1$ for which $F(f)$ is of first category. [This, however, is an immediate consequence of an example due to Lusin and Privaloff, Ann. Sci. Ecole Norm. Sup. (3) 42 (1925),

143–191, pp. 157–159.] Two examples are given: (A) $f(z)$ is a Blaschke product whose zeros have every point of $|z|=1$ as limit point. [The sign of the numerators in the product should have been reversed. Since, in a footnote on page 143, the author refers to Bagemihl and Seidel, Proc. Nat. Acad. Sci. U.S.A. **39** (1953), 1068–1075; [MR **15**, 295] in a context which is not clear to the reviewer, he should have been aware of the fact that this example is given explicitly on page 1070 of that paper.] (B) $f(z)$ is a univalent function which maps $|z|<1$ onto a bounded simply connected region all of whose prime ends are of diameter greater than some fixed $\delta>0$ [the existence of such regions was established by F. Frankl, Mat. Sb. **38** (1931), no. 3–4, 66–69]. *W. Seidel.*

Collingwood, E. F. On a theorem of Eggleston concerning cluster sets. J. London Math. Soc. **30** (1955), 425–428.

Let $f(z)$ be meromorphic in $|z|<1$ and ζ be any point of $|z|=1$. The global, radial, and angular cluster sets of $f(z)$ at $z=\zeta$ are denoted by $C(f, \zeta)$, $C_\rho(f, \zeta)$, and $C_\Delta(f, \zeta)$, respectively, and the outer angular cluster set $C_{\mathfrak{A}}(f, \zeta)$ is defined by $C_{\mathfrak{A}}(f, \zeta)=\bigcup_\Delta C_\Delta(f, \zeta)$, taken over all Stolz angles Δ with vertex at ζ. The sets of points ζ on $|z|=1$ for which $C(f, \zeta)$, $C_\rho(f, \zeta)$, $C_{\mathfrak{A}}(f, \zeta)$ reduce to single points are denoted by $G(f)$, $D(f)$, $F(f)$, respectively. It is shown that any two sets $G(f)$, $D(f)$, $F(f)$ differ at most by a set of first category. This is an extension of Eggleston's theorem [see the preceding review]. The proof consists in reducing the general case to the bounded case, to which Eggleston's theorem is then applied. The author fails to note, however, that the possibility of this reduction is a trivial consequence of Theorem 6 of Bagemihl and Seidel, Proc. Nat. Acad. Sci. U.S.A. **39** (1953), 1068–1075 [MR **15**, 295]. Some immediate corollaries are also given. *W. Seidel* (Notre Dame, Ind.).

Collingwood, E. F. A theorem on certain classes of singularities defined by cluster sets. J. London Math. Soc. **30** (1955), 422–424.

Let $f(z)$ be meromorphic in $|z|<1$. With the same notation for the various types of cluster sets as in the preceding review, the sets of points ζ on $|z|=1$ for which $C(f, \zeta)$, $C_\rho(f, \zeta)$, and $C_\Delta(f, \zeta)$, for every Stolz angle Δ with vertex ζ, coincide with the whole complex plane are denoted by $W(f)$, $S(f)$, $I(f)$, respectively. It is proved that any two of these sets differ at most by a set of first category on $|z|=1$. Here again it should be noted that this result, as well as some further remarks of the author, are obvious corollaries of Theorems 2 and 6 of the paper of Bagemihl and Seidel cited in the preceding review. *W. Seidel* (Notre Dame, Ind.).

Tsuji, Masatsugu. Function of U-class and its applications. J. Math. Soc. Japan **7** (1955), 166–176.

Let $w=f(z)$ be regular in $|z|<1$, with $|f(z)|<1$ there. Then $\lim f(z)=f(e^{i\theta})$ exists for almost all θ in $0\leqq\theta<2\pi$ as $z\to e^{i\theta}$ within any Stolz angle with vertex at $e^{i\theta}$. If for almost all θ, $|f(e^{i\theta})|=1$, $f(z)$ is said to belong to class U. The theory of functions of class U is applied to give new proofs of earlier results of Tsuji's [Kodai Math. Sem. Rep. **1950**, 89–92; MR **13**, 125] on open Riemann surfaces with null boundary, [Jap. J. Math. **19** (1944), 139–154; MR **8**, 508] on algebroid functions, and [Proc. Imp. Acad. Tokyo **19** (1943), 60–65; MR **8**, 508] on cluster sets at non-isolated boundary points of meromorphic functions in arbitrary domains. *W. Seidel.*

Tsuji, Masatsugu. On the cluster set of a meromorphic function. Comment. Math. Univ. St. Paul. **4** (1955), 5–9.

The study of cluster sets at non-isolated boundary points of meromorphic functions in arbitrary domains is continued and earlier results are generalized [see the preceding review] . *W. Seidel* (Notre Dame, Ind.).

Hiong, King-Lai. Un théorème d'unicité relatif à la théorie des fonctions méromorphes. C. R. Acad. Sci. Paris **241** (1955), 1691–1693.

Let $E^{(k)}(a)$ denote the set of all points z such that $f^{(k)}(z)=a$. Nevanlinna [Le théorème de Picard-Borel ⋯, Gauthier-Villars, Paris, 1929] has shown that a meromorphic function f in the plane is completely determined by five sets $E^{(0)}(a_i)$. The author announces several related results such as the following. When f does not take the values 0 and ∞, then f is determined up to a constant by three sets $E^{(k)}(a_i)$, k fixed, $a_i\neq0$, ∞. *J. Korevaar.*

Hiong, King-Lai. Un théorème fondamental sur les fonctions méromorphes et leurs primitives. C. R. Acad. Sci. Paris **242** (1956), 53–55.

Yet another adaptation of Nevanlinna's second Theorem is announced. A typical inequality is, with the usual notation,

$$T(r, f)<N(r, f)+N\left(r, \frac{1}{f-a}\right)+N\left(r, \frac{1}{f^{(-k)}-b}\right)-N\left(r, \frac{1}{f'}\right)+S,$$

where $f^{(-k)}$ denotes the k-fold integral of f, which is assumed to be meromorphic. *W. K. Hayman* (Exeter).

Hiong, King-Lai. Sur les fonctions holomorphes dans le cercle-unité admettant un ensemble de valeurs déficientes. J. Math. Pures Appl. (9) **34** (1955), 303–335.

Several rather complicated results are proved, of which the following is the simplest. A value α is said to be uniformly defective, of defect $\geqq\delta(\alpha)$, for a family of meromorphic functions in the unit disc when for every $\varepsilon>0$ there exists an $r_0<1$ such that

$$m\{r, 1/(f-\alpha)\}>\{\delta(\alpha)-\varepsilon\}\ T(r, f)$$

for every $r>r_0$ and every member f of the family. Now consider a family (f) as follows. (i) Every function f is regular in $|z|=r<1$ and of the form

$$f(z)=1+c_hz^h+\cdots\quad (h \text{ fixed}, |c_h|\geqq\gamma>0).$$

(ii) The family has a collection of q distinct uniformly defective values $\alpha_k\neq1$ of total defect $\geqq L>1$. The author shows that such a family is normal in the unit disc:

$$\log|f(z)|<A(\alpha_k, L, \gamma)(1-r)^{-1}\log\{2/(1-r)\},\ r<1,$$

for every f of the family. Related results include a theorem in which the condition $|c_h|\geqq\gamma>0$ is replaced by a different condition and a theorem where a different definition of defective value is used. *J. Korevaar* (Madison, Wis.).

Hiong, King-Lai. Sur les fonctions holomorphes dont les dérivées admettent une valeur exceptionnelle. Ann. Sci. Ecole Norm. Sup. (3) **72** (1955), 165–197.

The results of this paper differ from those of the paper reviewed above in two respects. Defective values are replaced by B-exceptional values, and in some of the results one of the B-exceptional values may belong to a derivative of f rather than to f itself. The simplest result of the latter kind will be cited. A value α is said to be uniformly B-exceptional for a family of meromorphic

functions in the unit disc when (a) there are fixed numbers $r_0<1$ and τ such that

$$N\{r, 1/(f-\alpha)\}<\tau \log \{1/(1-r)\}$$

for every $r>r_0$ and every member f of the family; (b) the numbers $n\{r, 1/(f-\alpha)\}$ are bounded by a finite function of r which is independent of f. Now consider a family (f) as follows. (i) Every function f is regular in $|z|=r<1$ and of the form

$$f(z)=c_0+\cdots+c_kz^k+\cdots \quad (k \text{ fixed}, c_0, c_k\neq 0, |c_k|\neq 1).$$

(ii) No f vanishes for $r<1$. (iii) The family $(f^{(k)})$ has the number 1 as a uniformly B-exceptional value. The author shows that such a family is normal in the unit disc.

The present paper also serves as a replacement for part of a previous paper of the author [same Ann. (3) **70** (1953), 149–180; MR **15**, 412]. A certain difficulty which arose in that paper and which was signalled by the reviewer is avoided in the two papers reviewed here by the introduction of the concepts of uniformly defective and uniformly exceptional values. *J. Korevaar.*

de La Vallée Poussin, C. Le théorème de Picard du point de vue topologique. Ann. Soc. Sci. Bruxelles. Sér. I. **69** (1955), 37–49.

The author studies Picard's (first) theorem from the topological point of view. For integral functions of the complex variable z omitting one finite value or meromorphic functions of z omitting two values he gives corresponding decompositions of the z-plane. Unfortunately the paper is obscure in a number of places. In particular, the proof of the Theorem of § 2, 7 appears incomplete. There may be infinitely many curves in the sector S which correspond to c but it may well be that none of them divides the sector S into two sectors. It does not seem to the reviewer that the properties of integral functions which the author uses explicitly are sufficient to insure the truth of Picard's theorem.

The author considers also more general types of exceptional values. According to the reviewer's interpretation an exceptional value in the general sense is simply what is usually called an asymptotic value. In this case the theorem of § 5, 22 is incorrect since there are integral functions which have arbitrarily many asymptotic values. Clearly also the proof of the theorem of § 5, 19 is incomplete. *J. A. Jenkins* (Notre Dame, Ind.).

Walsh, J. L. Détermination d'une fonction analytique par ses valeurs données dans une infinité dénombrable de points. Bull. Soc. Math. Belg. **1954**, 52–70 (1955).

Let D be a plane domain whose boundary consists of a finite number of Jordan curves, and let H be the class of functions $f(z)$ holomorphic in D and such that $\iint_D |f(z)|^2dS<\infty$. The paper is concerned with finding a function $f(z)\in H$ such that

$$(1) \qquad f(\alpha_k)=w_k \quad (k=1, 2, \cdots),$$

where $\alpha_k\in D$ and w_k are given.

An orthogonal system $\varphi_1, \varphi_2, \cdots$ with respect to the metric $(f, g)=\iint_D f(z)\overline{g(z)}dS$ is first constructed as follows: $\varphi_n(z)$ is the unique function which possesses the smallest norm among all functions $f(z)\in H$ for which

$$(2) \qquad f(\alpha_1)=\cdots=f(\alpha_{n-1})=0, \; f(\alpha_n)=1.$$

With the help of this system an arbitrary $f(z)\in H$ is developed into an orthogonal series

$$f(z)\sim a_1\varphi_1(z)+a_2\varphi_2(z)+\cdots \quad (a_n=(f, \varphi_n)/(\varphi_n, \varphi_n))$$

as well as into an interpolation series

$$f(z)\sim b_1\varphi_1(z)+b_2\varphi_2(z)+\cdots.$$

The coefficients b_n are obtained by regarding the above relation as an equality, putting successively $z=\alpha_1$, $z=\alpha_2$, $\cdots$, and taking into account the equations (2).

The author proves that for every $f(z)\in H$ these two developments are identical. Introducing the normalized functions $\psi_n=\varphi_n/(\varphi_n, \varphi_n)$ and writing $f(z)\sim c_1\psi_1(z)+c_2\psi_2(z)+\cdots$ $(c_n=(f, \psi_n))$, he thus obtains the main result of the paper: There exists an $f(z)\in H$ satisfying (1) if and only if $\sum |c_n|^2<\infty$.

At the end of the paper, after studying the uniqueness of the solution, the author lists a number of unsolved problems which are similar to the above. *O. Lehto.*

Grunsky, Helmut. Eine funktionentheoretische Integralformel. Math. Z. **63** (1955), 320–323.

Let $f(z)$ be a continuously differentiable function of the complex variable $z=x+iy$ in a domain $\mathfrak{G}$, and let $\mathfrak{C}\subset\mathfrak{G}$ be a compact set whose boundary C consists of a finite number of simple rectifiable Jordan curves. Using complex notation, the author first gives a simple proof for the validity of the relation

$$(1) \qquad 2i\iint_{\mathfrak{C}} \frac{\partial f}{\partial \bar{z}}(dz)=\int_C f(z)dz \quad ((dz)=\text{element of area}),$$

which is a complex form of the classical formulas of Gauss and Stokes.

The author then makes the more special assumptions that $f(z)$ is analytic and $\mathfrak{C}$ a polygonal domain $\mathfrak{P}$ with boundary P and vertices $z_1, z_2, \cdots, z_n$, and shows that in this case a further integration can be carried out in (1). Writing $f'(\bar{z})$ instead of $f(z)$ and assuming that $f(z)$ is analytic in $\mathfrak{P}$ and continuous on the boundary, he obtains the formula

$$(2) \qquad \iint_{\mathfrak{P}} f''(z)(dz)=\sum_{\nu=1}^{n} \delta_\nu f(z_\nu).$$

Here the coefficients δ_ν, which depend on $\mathfrak{P}$ only, can be explicitly expressed. At the end of the paper, certain applications of (2) are briefly referred to. *O. Lehto.*

Paatero, V. Über die Randdrehung der mehrblättrigen einfach zusammenhängenden Gebiete. Ann. Acad. Sci. Fenn. Ser. A. I. no. **194** (1955), 7 pp.

Continuing his work on boundary-rotation [same Ann. 33 (1931), no. 9; cf. ibid. no. **128** (1952); MR **14**, 861], the author proves that this notion can be interpreted in two ways also in case of domains which are situated on a Riemann surface and contain a finite number of branch-points. *Y. Komatu* (Tokyo).

Jabotinsky, E. Iterational invariants. Technion. Israel Inst. Tech. Sci. Publ. **6** (1954/5), 64–80. (Hebrew summary)

Let $F(z)=F_1(z)=\sum_1^\infty a_nz^n$ for $|z|<\varrho$ (>0) and let $F_m(z)=F(F_{m-1}(z))=\sum_1^\infty a_{nm}z^n$. The question of the existence of "iterational invariants" or functions $\phi_n=\phi(a_{1m}, a_{2m}, \cdots, a_{nm})$ independent of m was raised by Hadamard [Bull. Amer. Math. Soc. **50** (1944), 67–75; MR **5**, 185]. In the present paper it is shown that for $|a_1|\neq 1$ the unique solution $G(z)=\sum_1^\infty g_nz^n$ of the equation $G(F(z))=a_1G(z)$ provides such a set of invariants in $g_2, g_3, g_4, \cdots$. Similarly $G(z)/G'(z)=\sum_1^\infty j_nz^n$ provides another set coinciding with those given by Hadamard. These coefficients are invariant with respect to iteration of fractional

or complex order. Define $f_{p,q}(m)$ by $\{F_m(z)\}^p=\sum_p^\infty f_{p,q}(m)z^q$. It is shown that $f_{p,q}(m)=\sum_{s=p}^{s=q} H_{p,s,q}(a_1^m)^s$ and a number of relations generalising those involved in Lagrange's expansion are given. An explicit example based on $G_{-1}(z)=ze^{-z}$ is elaborated. Some partial results are derived when $|a_1|=1$, the cases $a_1=1$ and a_1 a root of unity requiring separate treatment.

The principle of equating coefficients is used extensively and the paper is to be noted as giving by implication an elementary account of various results in the theory of Koenig's function and analytic iteration.

A. J. Macintyre (Aberdeen).

Mikusiński, J. On Dirichlet series with complex exponents. Ann. Polon. Math. **2** (1955), 254–256 (1956).

In this paper the author is concerned with series of the form $\sum a_m e^{-\beta_n s}$ where the coefficient sequence $\{a_n\}$ and the exponent sequence $\{\beta_n\}$ are complex numbers subject to the limitation $\lim (\beta_n'/\log n)=\infty$ and where $\beta_n'=\mathrm{Re}\,(\beta_n)$. The classical theory of Dirichlet series is concerned with such series for the case in which the sequence $\{\beta_n\}$ is a sequence of positive real numbers strictly increasing to infinity. Since there are no references we assume the author is unaware that this problem has been studied previously. Hille [Ann. of Math. (2) **25** (1924), 261–278] and Ritt [Trans. Amer. Math. Soc. **18** (1917), 27–49] have each proven theorems which would seem to include those of the author. Also the author states that the region of convergence of $\sum a_n e^{-\beta_n s}$ is convex. However Schnee [Thesis, Berlin, 1908] has given an example of such a series with an isolated point of convergence. The author shows (as did Hille) that the region of absolute convergence is a convex set. If this set is two-dimensional then the series represents a single-valued analytic function in the region.

V. F. Cowling (Lexington, Ky.).

Rose, Donald Clayton. On general Dirichlet series. Duke Math. J. **23** (1956), 73–82.

Let $f(s)=\sum_{n=1}^\infty a_n e^{-\lambda_n s}$ $(s=\sigma+i\tau)$ be a general Dirichlet series with finite abscissa of convergence. The author studies the problem of analytic extension of the function $f(s)$. The Lindelöf methods [Le calcul des résidus, Gauthier-Villars, Paris, 1905] are used and a general theorem and two sets of sufficient conditions that the analytic extension of $f(s)$ exist, are established. The whole paper generalizes the results of V. F. Cowling [Duke Math. J. **14** (1947), 907–911; MR **9**, 276] and the reviewer [Rev. Mat. Hisp.-Amer. (4) **13** (1953), 312–319; MR **15**, 699].

A typical result obtained is as follows: Let $\psi(u)$ and $\phi(u)$ be two real rectifiable functions defined for $u\geqq A>0$ (A non integral) bounded away from zero and positive for $u>A+\varepsilon$ and vanishing for $u=A$, and let D be the simply connected set in the plane $w=u+iv$ given by $u\geqq A$, $\phi(u)\geqq v\geqq -\psi(u)$, $u<\infty$. Suppose there exist two functions $a(w)$, $\lambda(w)$ analytic in D, such that $a(n)=a_n$, $\lambda(n)=\lambda_n$ for $n=[A]$ (n integral), and that the following set of conditions holds:

$$(1+\theta_1)\log u/\psi(u)\leqq k_1,\ (1+\theta_2)\log u/\phi(u)\leqq k_2,$$

$$\log[(1+(\psi'(u))^2]^{1/2}/\psi(u)\leqq l_1,\ \log[1+(\phi'(u))^2]^{1/2}/\phi(u)\leqq l_2$$

for some positive θ_1, θ_2 and all sufficiently large u, where $0\leqq k_1+l_1<L$, $0\leqq k_2+l_2<2\pi-L$, and $|a(w)|=O\{\exp[PH(u)-Lv]\}$ for some real P and all sufficiently large u, where $0<H(u)=\mathrm{Re}\,[\lambda(w)]$ and $H(u)\to\infty$ as $u\to\infty$. Then $f(s)$ can be analytically extended to any closed bounded domain contained in the region defined by:

$$(\sigma-P)H(u)/\psi(u)-\tau T[u,-\psi(u)]/\psi(u)\geqq -h_1$$

and

$$(\sigma-P)H(u)/\phi(u)-\tau T[u,\phi(u)]/\phi(u)\geqq -h_2,$$

where h_1, h_2 are positive constants and $T(u,v)=\mathrm{Im}[\lambda(u+iv)]$.

A. G. Azpeitia (Providence, R.I.).

Siegel, Carl Ludwig. Die Funktionalgleichungen einiger Dirichletscher Reihen. Math. Z. **63** (1956), 363–373.

Hecke has introduced into the theory of Eisenstein series a convergence factor $|c\tau-a|^{-z}$, as in

$$\Phi_m(\tau,z)=1+\sum\gamma^m\left(\frac{a}{c}\right)(c\tau-a)^{-m/2}|c\tau-a|^{-z},$$

which by analytic continuation reduces to the theta-functions $\Phi_m(\tau,0)=\vartheta_m(\tau,0)$ when $z=0$ $(m=1,3,4)$. Considered as a function of z, $\Phi_m(z)$ is meromorphic and satisfies a simple functional equation.

Generalizing this situation the author defines

$$\varphi(\tau,z)=\eta^{z/2}\sum Q(M_r,\tau)\vartheta(\infty)|c\tau-a|^{-z}\quad(\tau=\xi+i\eta),$$

where Q is the multiplier in the transformation law of $\vartheta(\tau)$ and M_r runs over a set of inequivalent modular substitutions. Siegel proves that $\varphi(\tau,0)=\vartheta(\tau)$, that φ satisfies the same transformation equation (in τ) as $\vartheta(\tau)$ does, and satisfies also the functional equation

$$\psi(\tau,z)=\psi(\tau,1-z),$$

where $\psi=\varphi(\tau,z)\cdot(2^{z/2}+2^{-z/2})\pi^{-z}\Gamma(z)\zeta(2z)$, ζ being the Riemann zeta-function.

At the end of the paper Siegel generalizes Riemann's second proof of the functional equation for $\zeta(z)$ (the one which uses ϑ-functions) by setting up an analogous integral $\varrho(s,z)$. We have $\varrho(s,0)=\zeta(s)$, and $\varrho(s,z)$ is developed as an Eisenstein series. The author proposes the problem of discussing the zeros of $\varrho(s,z)$ on the line $\Re s=\frac{1}{2}$.

J. Lehner (Los Alamos, N.M.).

Siegel, Carl Ludwig. Zur Theorie der Modulfunktionen n-ten Grades. Comm. Pure Appl. Math. **8** (1955), 677–681.

In the theory of automorphic functions of one variable, the fundamental region F of a Fuchsian group of the first kind (limit-circle group) can be compactified by adjoining the parabolic boundary points (if any). In the case of the modular group one adds the infinite point ($i\infty$) of F, or equivalently the point $q=0$ of the corresponding local variable $q=e^{2\pi i z}$.

In this paper Siegel accomplishes the compactification of the fundamental region of the matrix modular group of degree n. Let $Z=X+iY$, $Y>0$, where X, Y are real symmetric matrices of order n. Siegel proves the existence of ν $(\nu=\frac{1}{2}n(n+1))$ variables $q_1,\cdots,q_\nu$ such that when $Z\to\infty$ $(y_{nn}\to\infty)$ there is a subsequence on which one of the q's$\to0$ while the others tend to limits lying in $|q_k|<1$. The q's and z's are related by a reversible analytic transformation. F is then made compact by adding the finite number of surfaces $q_k=0$ $(k=1,2,\cdots,\nu)$. In the proof decisive use is made of the arithmetic properties of the modular group via Minkowski's reduction theory of positive quadratic forms.

The case $n=2$ is worked out explicitly. Set $z_{11}=z_1$, $z_{12}=z_2$, $z_{22}=z_3$, and $q_k=\exp 2\pi i w_k$ $(k=1,2,3)$, with $w_1=z_1-2z_2$, $w_2=z_2$, $w_3=z_3-z_1$. By the reduction theory we have $0\leqq 2y_2\leqq y_1\leqq y_3$. When $y_3\to\infty$, it is immediately

verified that there is a subsequence on which the above requirements on the q's are satisfied. The local variable is shown to be $q_1q_2^2q_3$. *J. Lehner* (Los Alamos, N.M.).

Becker, Hugo. Poincarésche Reihen zur hermitischen Modulgruppe. Math. Ann. **129** (1955), 187–208.

The author studies Poincaré series associated with certain discontinuous subgroups $G^{(n)}$ of the hermitian modular group $H^{(n)}$ of degree n [for definitions and notation see Braun, Ann. of Math. (2) **53** (1951), 143–160; MR **12**, 482]. Let $G^{(n)}$ be a congruence subgroup modulo q over an imaginary quadratic field K. The Poincaré series

$$g_{-k}(Z, F)=\sum_{\sigma\in V(f)} e^{2\pi i \mathrm{Sp}(\sigma F(Z))}v(\sigma)^{-1}|C_\sigma Z+D_\sigma|^{-k}$$

is shown to converge absolutely in Z when $k>\min(4n-2, 2(n+s))$, $k\equiv O(w)$, and to represent a modular form of dimension $-k$ belonging to $G^{(n)}$ and having multipliers $v(\sigma)$ ($|v(\sigma)|=1$). Here F is an "exponent matrix" of rank s, $V(F)$ runs over a set of inequivalent substitutions of $G^{(n)}$, and w is the number of roots of unity in K. For the modular groups $H^{(n)}$ the author establishes a lower bound for the number of linearly independent modular forms of dimension $-k$ (provided $k>4n-2$, $k\equiv O(w)$).
J. Lehner (Los Alamos, N.M.).

Huber, Heinz. Über eine neue Klasse automorpher Funktionen und ein Gitterpunktproblem in der hyperbolischen Ebene. I. Comment. Math. Helv. **30** (1956), 20–62 (1955).

Let Γ be a discontinuous group of translations of the hyperbolic plane H on itself, and let H mod Γ be compact. Then p (genus) of H mod Γ is >1. Let $\mathfrak{R}$ be an infinite class of conjugate elements of Γ. Denote by $N_{\mathfrak{R}}(z, t)$ the number of elements T such that $T\in\mathfrak{R}$ and $\varrho(z, Tz)\leq t$, where $z\in H$ and ϱ is the hyperbolic distance. The author proves:

$$(1)\qquad N_R(z, t)\sim\frac{1}{4\pi(p-1)}\,\frac{1}{\nu(\mathfrak{R})}\,\frac{\mu(\mathfrak{R})}{\sin\frac{1}{2}\mu(\mathfrak{R})}\,e^{t/2},$$

an estimate which is independent of z. In this formula $\nu=|r|$, where $T=P^r$ (and P is not a power of another element), $T\in\mathfrak{R}$; $\mu(\mathfrak{R})=\inf\varrho(z, Tz)$ for $z\in H$, $T\in\mathfrak{R}$. The result is obtained by studying the Dirichlet series $\sum_{T\in\mathfrak{R}}(\cos\varrho(z, Tz)-1)^{-s}$. The series converges for $\Re s>\frac{1}{2}$ but can be continued over the whole s-plane, and possesses a representation which exhibits its asymptotic behavior in s. Application of the Wiener-Ikehara Tauberian theorem then yields (1). *J. Lehner* (Los Alamos, N.M.).

Myrberg, P. J. Darstellung automorpher Funktionen durch Zusammensetzung von elliptischen und fuchsoiden Funktionen. Ann. Acad. Sci. Fenn. Ser. A. I. no. **200** (1955), 9 pp.

The author proves that every automorphic function on a Fuchsian group Γ (defined on $|z|<1$) can be written as the quotient of two "automorphic theta-functions of the first order." Here Γ is the group derived from the uniformization of an algebraic Riemann surface of genus ≤ 2. By an automorphic theta function of order one the author means a function $u(z)$, entire in $|z|<1$, such that the logarithm of the multiplier of every function on Γ can be expressed as $c(S)u(z)+b(S)$, where S is a substitution of Γ. The automorphic theta functions in turn are constructed from ordinary elliptic theta functions and certain fuchsoid functions (given explicitly) which belong to normal subgroups of Γ. The author shows that his method is not applicable in general when the genus of the Riemann surface exceeds two. *J. Lehner* (Los Alamos, N.M.).

Fréchet, Maurice. The para-analytic functions in n dimensions. J. Reine Angew. Math. **195** (1956), 22–41 (1955). (Esperanto. French summary)

The author proves the results announced in his two previous papers [C.R. Acad. Sci. Paris **236** (1953), 1832–1834, 2191–2193; MR **15**, 117, 416]. Para-analytic function is defined in a linear, associative, commutative algebra with a unit element as a generalization of the analytic function in complex function theory. Various properties of these functions are developed. The generalized Cauchy-Riemann differential equations (29) and the generalized Laplacian equations (33) are the same as those derived by R. D. Wagner [Duke Math. J. **15** (1948), 455–461; MR **10**, 30]. *J. A. Ward* (Holloman, N.M.).

Roşculeţ, Marcel N. Fonctions polygènes dans les algèbres linéaires associatives et commutatives. C. R. Acad. Sci. Paris **242** (1956), 51–52.

For the derivative of a polygenic function in a linear algebra the author obtains the hypersurfaces given by P. W. Ketchum and W. T. Martin [Bull. Amer. Math. Soc. **38** (1932), 66–72]. He then uses an exponential form to obtain higher derivatives. *J. A. Ward.*

See also: Schoenberg, p. 591; Gaier, p. 604; Sikkema, p. 605; Akutowicz, p. 609; Thron, p. 631; Ahlfors, p. 657, 668; Blanch and Jackson, p. 669; Fuchs, p. 695; Shapiro, p. 695.

Harmonic Functions, Potential Theory

Brelot, M. Etude et extensions du principe de Dirichlet. Ann. Inst. Fourier, Grenoble **5** (1953–1954), 371–419 (1955).

The classical form of the Dirichlet principle asserts that among all functions in a domain with prescribed boundary values the harmonic one has the smallest Dirichlet integral. This is equivalent to the possibility of splitting up a given function into the sum of a harmonic function and one which is zero on the boundary. This paper contains a definitive study of the question: how wide a class of functions, boundary values, and domains can be admitted. The tools are the theory of the Dirichlet problem of subharmonic functions, potential theory, and a refined real variable theory. *P. D. Lax.*

Tsuji, Masatsugu. A simple proof of Dirichlet principle. J. Math. Soc. Japan **7** (1955), 67–75.

L'A. traite le principe de Dirichlet en s'appuyant sur la solution (indépendante) de Perron du problème de Dirichlet. Il ne considère qu'un domaine du plan ou d'une surface de Riemann, avec une frontière formée d'un nombre fini de courbes de Jordan, et des fonctions à gradient continu par morceaux. Les démonstrations utilisent la représentation conforme de régions annulaires. Remarquons que l'idée de départ est aussi celle du Ref. dans des travaux récents [C.R. Acad. Sci. Paris **235** (1952), 598–600; MR **16**, 35; et l'oeuvre analysé ci-dessus; voir aussi Deny et Lions, Ann. Inst. Fourier, Grenoble **5** (1953–54), 305–370; MR **17**, 646] mais les domaines (à un nombre quelconque de dimensions), les frontières et fonctions sont beaucoup plus généraux. *M. Brelot* (Paris).

Górski, J. Méthode des points extrémaux de résolution du problème de Dirichlet dans l'espace. Ann. Polon. Math. 1 (1955), 418–429.

The method, from transfinite diameter theory, is applied to minimise the energy of a distribution, over the boundary F of an infinite domain D, of unit positive charge in the presence of an external potential $-\lambda f$ (f continuous, λ a parameter), under the condition that the distribution is of constant sign. [Two-fluid theory with only one fluid present: the quasi-physical description is the reviewer's.] The author proves that an extremal distribution exists, as a non-negative additive function of sets μ_λ, carried by a subset F_λ of F, and that the resulting potential is constant on F_λ save for a set of capacity zero. He builds up, from μ_λ and a free distribution on F_λ, the solution of Dirichlet's problem for the domain D_λ exterior to F_λ with the boundary values f on F_λ. Finally he shows that $F_\lambda \to F_0$ as $\lambda \to 0$, where F_0 has the same capacity as F.
H. D. Ursell (Leeds).

Lelong-Ferrand, Jacqueline. Sur la décomposition spectrale des formes harmoniques. C. R. Acad. Sci. Paris 242 (1956), 600–602.

Suppose V^n is a metrically complete, orientable, C^∞ Riemannian manifold. Assume that the curvature of V^n is bounded; then this note discusses the behavior of solutions of $\Delta\varphi^p+\lambda\varphi^p=0$. In particular, the author is concerned with the rate of increase of the solutions of this equation. Now on a Riemannian manifold φ^p is to denote a differential form of degree p and Δ denotes the Laplacian operator [G. de Rham, Variétés différentiables, Hermann, Paris, 1955; MR 16, 957]. We let $|\varphi^p|^2$ denote the square of the scalar product of φ^p with itself. The first result is that if we let r denote distance from a fixed point on V^n, then the set of values λ admitting a solution of $\Delta\varphi^p+\lambda\varphi^p=0$ which is non-trivial and which satisfies $|\varphi^p|<e^{\alpha r}$, has an upper bound of the form $\alpha^2+B\alpha+C$, where B and C are constants. That is to say, if we impose an exponential growth condition, then the set of eigenvalues admitting an eigenform which satisfies the growth condition is bounded above.

Next the author considers V^n to be locally reducible; that is, V^n is given an integrable local-product structure and a Riemannian metric which may be expressed as

$$ds^2=g_{ij}(x)dx^idx^j+h_{\alpha\beta}(y)dy^\alpha dy^\beta \quad (i, j=1, \cdots, m; \; \alpha, \beta=1,\cdots, m')$$

with $m+m'=n$. In this case it is stated that if V^n is a metrically complete, orientable, Riemannian manifold which is locally reducible and has bounded curvature and if for the isometric covering $X\times Y$ which V^n will admit [de Rham, Comment. Math. Helv. 26 (1952), 328–344; MR 14, 584] either X or Y is compact, then every solution of $\Delta\varphi^p+\lambda\varphi^p=0$ which admits an exponential growth bound may be expressed as $\varphi^p=\sum_{r=1}^R \psi_r(x)\wedge\eta_r(y)$.
P. E. Conner (Princeton, N.J.).

Fichera, Gaetano. Sull'esistenza delle forme differenziali armoniche. Rend. Sem. Mat. Univ. Padova 24 (1955), 523–545.

This paper contains what appears to be a straight forward and relatively simple proof of the theorem of Hodge that there exist B_k linearly independent harmonic k-forms on a compact differentiable manifold, B_k being the kth Betti number. The proof is based on an explicit formula giving solutions for the boundary-value problem $\mathcal{E}^{2m}u=\phi$ in euclidean n-space. Here u and ϕ are k-forms, ϕ having square summable coefficients, and $\mathcal{E}$ is the harmonic differential operator. The boundary conditions are the vanishing of the coefficients of u and their first derivatives. The solution formula is $u=T^{2m}\phi-T^mPT^m\phi$, where T is defined by means of a kernel $s(x, y)$ which is a "fundamental solution" of $\mathcal{E}u=0$, and P is a projection. The integer m is ultimately taken greater than $n/4+2$ to ensure that u is of class C^2. There is a fairly complete bibliography on harmonic forms which, however, does not include de Rham's recent book, Variétés différentiables [Hermann, Paris, 1955; MR 16, 957].
P. A. Smith.

★**Cimmino, G. Spazi hilbertiani di funzioni armoniche e questioni connesse.** Convegno Internazionale sulle Equazioni Lineari alle Derivate Parziali, Trieste, 1954, pp. 76–85. Edizioni Cremonese, Roma, 1955. 3000 Lire.

Remarques bibliographiques sur l'emploi de diverses normes hilbertiennes pour résoudre les problèmes de contour relatifs à l'équation de Laplace.
J. Deny.

Gabriel, R. M. An inequality concerning integrals of 3-dimensional harmonic functions. J. London Math. Soc. 31 (1956), 79–82.

Utilisant et complétant deux notes [même J. 24 (1949), 154–156, 313–316; MR 11, 108, 435], l'auteur considère une fonction harmonique u dans un domaine de R^3 contenant deux surfaces convexes fermées emboitées. L'intégrale superficielle de $|u|^\lambda$ sur la plus petite est majorée par l'intégrale sur la grande, multipliée par $2^\lambda(\lambda/(\lambda-1))^\lambda$ $(1<\lambda\leq 2)$.
M. Brelot (Paris).

Jeffreys, Harold. Two properties of spherical harmonics. Quart. J. Mech. Appl. Math. 8 (1955), 448–451.

Let K_n be a solid harmonic of degree n, and let dS be the element of area of the sphere S of radius a. The author proves

$$\iint_S\left(\frac{\partial K_n}{\partial x_i}\right)^2 dS=\frac{n(2n+1)}{a^2}\iint_S K_n^2 dS$$

together with a corresponding formula for $\partial^2K_n/\partial x_i\partial x_k$, and gives an application to the theory of elasticity.

Defining the irregularity of a function f on the unit sphere Ω (with element of area $d\omega$) as

$$\iint_\Omega (\nabla_1 f)^2 d\omega \Big/ \iint_\Omega f^2 d\omega,$$

where $(\nabla_1 f)^2$ is Beltrami's first differential parameter (the square of the gradient on the sphere), he shows that the irregularity is stationary under small variations of f if f is a surface harmonic, and that the irregularity of any function is at least equal to that of the lowest term in its expansion in surface harmonics.
A. Erdélyi.

Zoller, Konrad. Das Newtonsche Potential einer Kreisfläche. Z. Angew. Math. Mech. 35 (1955), 475–476.

See also: De Giorgi, p. 596; Cesari, p. 596; Rubinstein, p. 625; Rizza, p. 662; Grauert et Remmert, p. 662; Lelong, p. 662; Ludford, Martinek and Yeh, p. 678.

Series, Summability

Gaier, Dieter. On modified Borel methods. Proc. Amer. Math. Soc. 6 (1955), 873–879.

Borel's two methods of summability [Hardy, Divergent

series, Oxford, 1949, ch. 8; MR 11, 25] depend on a parameter x which tends to infinity through all positive real values. They are here considered when x is allowed to tend to infinity by integer values only. The usual properties and mutual relations of the two methods no longer hold unless the series $\sum a_n$ to be summed satisfies some special condition such as $a_n=O(K^n)$ with $K<(\pi^2+1)^{\frac{1}{2}}$. The analysis depends on Cartwright's theorem on integral functions bounded at a sequence of points and its generalisations [Boas, Entire functions, Academic Press, New York, 1954, ch. 10; MR 16, 914]. Similar considerations applied to Doetsch's "Cesàro-Borel" methods [Hardy, loc. cit., pp. 237–238] lead to problems on integral functions at present unsolved. *A. J. Macintyre.*

Korenblyum, B. I. On the asymptotic behavior of Laplace integrals near the boundary of a region of convergence. Dokl. Akad. Nauk SSSR (N.S.) 104 (1955), 173–176.

If $f(x)=\sum_0^\infty a_nx^n$, $g(x)=\sum_0^\infty b_nx^n$ converge for $|x|<1$, $S_n=\sum_0^n a_k$, $\sigma_n=\sum_0^n b_k$ and $f(x)\sim g(x)$ for $x\to 1-$, then $S_n\sim\sigma_n$ for $n\to\infty$, provided (a) $\sigma_m/\sigma_n\to 1$ for $n\to\infty$, $m/n\to 1$ and (b) $a_n\geqq 0$, $b_n\geqq 0$. There is a corresponding result for Laplace integrals. Karamata [J. Reine Angew. Math. 164 (1931), 27–39], whose technique the author follows, proved similar theorems for σ_n satisfying a condition (of "regular growth") more stringent than (a), but without the assumption (b). *G. G. Lorentz.*

Sikkema, P. C. On linear recursion formulae. Nederl. Akad. Wetensch. Proc. Ser. A. 58=Indag. Math. 17 (1955), 596–607.

Let $\alpha(z)=1+\sum_1^\infty a_nz^n$ and let $\beta(z)$ be its formal reciprocal $1/\alpha(z)=\beta(z)=1+\sum_1^\infty b_nz^n$. If $\limsup|a_n|^{1/n}=\infty$, then $\limsup|b_n|^{1/n}=\infty$. The author is concerned with more refined growth estimates of the sequences $\{a_n\}$ and $\{b_n\}$. The following is typical: let $\limsup\{(n!)^{-\lambda}|a_n|\}^{1/n}=\nu<\infty$, with $\lambda>0$. Then, $\limsup\{(n!)^{-\lambda}|b_n|\}^{1/n}=\nu$. *R. C. Buck* (Madison, Wis.).

König, Bedřich. Calculation of the sum of series. Časopis Pěst. Mat. 80 (1955), 191–201. (Czech)

Verfasser setzt unter geeigneten Voraussetzungen

$$\sum_{k=k_0}^{n}\frac{1}{f(k)}=A_{-1}\left[\frac{1}{(k_0-\frac12)^{s-1}}-\frac{1}{(n+\frac12)^{s-1}}\right]+\cdots+A_d\left[\frac{1}{(k_0-\frac12)^{s+d}}-\frac{1}{(n+\frac12)^{s+d}}\right]+R$$

wobei die A_i (unabhängig von d und n) leicht aus der Entwicklung

$$\frac{1}{f(z)}=b_0z^{-s}+b_1z^{-s-1}+\cdots$$

zu bestimmen sind. Ferner gibt Verfasser Darstellungen und Abschätzungen des Restes. Als Beispiel berechnet er $\zeta(2)=\sum k^{-2}$ auf 25 Stellen, sowie

$$\sum_{k=10}^{n}(4k^4-k^2+1)^{\frac12}\text{ und }\sum_{k=20}^{\infty}e^{k\pi i/3}/(4k^4-k^2+1)^{\frac12}.$$

K. Zeller (Tübingen).

Braun, Günther. Zur Methode der stationären Phase. Acta Phys. Austriaca 10 (1956), 8–33.

An asymptotic expansion for real $\omega\to\infty$ is given for $\int_a^b g(x)e^{i\omega f(x)}dx$, assuming that $f(x)$ is real, $f(x)$ and $g(x)$ are sufficiently differentiable, and $f'(x)\neq 0$ in the finite interval (a,b). It is also assumed that $f(x)$ has stationary points of integer orders at a and b while $g(x)$ and its derivatives have finite limits at a and b. [A similar result, but one in which $f(x)$ can have stationary points of fractional order at a and b while $g(x)$ can have certain singularities at a and b, has been given by A. Erdélyi, J. Soc. Indust. Math. 3 (1955), 17–27; MR 17, 29.] The result is used to derive an analogous expansion for $\iint g(x,y)e^{i\omega f(x,y)}dxdy$. *T. E. Hull* (Vancouver, B.C.).

Schulze, Herbert. Über die Reihenentwicklung des Ausdruckes $\alpha^n+\beta^n$. Z. Angew. Math. Mech. 35 (1955), 462–463.

See also: Hartman, p. 586; Mikusiński, p. 602; Rose, p. 602; Keogh, p. 609; Sinvhal, p. 609; Ganelius, p. 609.

Interpolation, Approximation, Orthogonal Functions

Nöbeling, Georg; und Bauer, Heinz. Allgemeine Approximationskriterien mit Anwendungen. Jber. Deutsch. Math. Verein. 58 (1955), Abt. 1, 54–72.

The authors give a series of criteria for the uniform approximability of a function f by bounded real functions of a class V, generally restricted to be a vector lattice satisfying the Stone axiom: $f\in V\sim\min(1,f)\in V$. The results seem to be detailed anatomizations of the theme and proof of the now classical Stone-Weierstrass theorem. Topology in the domain E of the functions, hence also continuity of the functions, do not enter the main discussion, and occasionally the range of each function is permitted to be a subset of a locally convex topological vector space R. For this latter case the criterion has the following form: Let A be the set of mappings from E into R of the form $\sum_1^n f_k(x)w_k$, n arbitrary, finite, $f_k(x)\in V$, $w_k\in R$. Then a mapping f of E into R is uniformly approximable from A if and only if (a) for each neighborhood U of 0 in R, there is a $\delta>0$ and a finite set $f_1, f_2, \cdots, f_m$ such that $|f_k(y)-f_k(z)|<\delta$ for all k implies $f(y)-f(z)\in U$ and $|f_k(x)|<\delta$ for all k implies $f(x)\in U$.

Numerous applications of the criteria are given to the theory of almost periodic functions on groups, to the Weil theory of the associated compact group of a group, and to new forms of the Stone-Weierstrass theorem. Included also is the following criterion: Let W be a set of bounded real functions on E. Then E is susceptible to a uniform structure relative to which the functions of W are the totality of uniformly continuous functions, if and only if (a) W is an algebra and $1\in W$, (b) W is complete in the topology of uniform convergence. *B. Gelbaum.*

★ **Rosenbloom, P. C.; and Warschawski, S. E. Approximation by polynomials.** Lectures on functions of a complex variable, pp. 287–302. The University of Michigan Press, Ann Arbor, 1955. $10.00.

Have

The authors obtain various quantitative results on the degree of approximation by polynomials. They state that these results will be applied to variational problems. Proofs in the paper are omitted or briefly sketched. Let C be a rectifiable Jordan curve, D its interior and D_1 its exterior. Denote by ψ the conformal map of $|w|>1$ onto D_1 and denote by ϕ the inverse map. Here $\psi(w)=aw+\cdots(a>0)$. For $n=1, 2, \cdots$ the Faber polynomial $\mathcal{F}_n(z)$ is then defined as the polynomial part of the expansion of $\phi^n\cdot(\phi')^{\frac12}$ about $z=\infty$. The authors consider the class $H(2,D)$ of functions analytic in D with $\int_{C_n}|f(z)|^2|dz|$ bounded for a sequence of curves C_n ap-

proaching C. They define the "Faber coefficients" $\{a_\nu\}$ of f, where $\nu=0, \pm 1, \pm 2, \cdots$, for f in $H(2, D)$ as the corresponding Fourier coefficients of the function $f(\psi(w))\cdot(\psi'(w))^{\frac{1}{2}}$ on $|w|=1$. Under a regularity condition on C, they show that except for a set of measure 0 on C which is independent of f the "Faber expansion" $\sum_{\nu=0}^{\infty} a_\nu \mathcal{F}_\nu(z)$ of f on C and the Fourier series $\sum_{\nu=0}^{\infty} a_\nu w^\nu$ on $|w|=1$ are equiconvergent. Further, $\sum_0^\infty a_\nu \mathcal{F}_\nu$ converges to f in L^2-norm on C and converges uniformly on closed subsets of D. In fact

$$\|f-\sum_{\nu=0}^{n} a_\nu \mathcal{F}_\nu\|^2 \leqq \text{constant} \times \sum_{n+1}^{\infty} |a_\nu|^2.$$

The authors next consider the orthogonal Szegö polynomials on C, $p_0, p_1, \cdots$. For f in $H(2, D)$ set

$$c_n=\frac{1}{L}\int_C f\bar{p}_n|dz|,\quad \sigma_N=\sum_0^N c_n p_n,\ s_N=\sum_0^N a_n \mathcal{F}_n,$$

where L denotes the length of C. Then

$$\sum_{\nu=n+1}^{\infty} |a_\nu|^2 \leqq \|f-\sigma_n\|^2 \leqq \|f-s_n\|^2 \leqq (1+o(1)) \sum_{n+1}^{\infty} |a_\nu|^2.$$

Further, on the complement of a set of small measure on C the expansions in Szegö and Faber polynomials are uniformly equiconvergent. Let $K(z, \bar{z}_0)=\sum_0^\infty p_n(z)\overline{p_n(z_0)}$ be the Szegö kernel for D. The authors give quantitative relations between the rapidity of convergence of this series and the boundary behavior of the function Φ, $\Phi(z_0)=0$, $\Phi(z_0)>0$, which maps D onto $|w|<1$.

Finally, the authors consider "Carleman polynomials" for D, i.e. the polynomials $q_0, q_1, \cdots$ obtained by orthogonalizing $1, z, z^2, \cdots$ with respect to the inner product $(f, g)=\int_D f(z)\overline{g(z)}dA$, where dA is the element of area. They generalize to a class of regions of sufficiently smooth boundary results of Carleman on the asymptotic behavior of q_n which he obtained for the case of an analytic boundary. *J. Wermer* (Providence, R.I.).

Quilghini, D. Sull'approssimazione delle funzioni continue di due variabili mediante polinomi di interpolazione algebrici e trigonometrici. Riv. Mat. Univ. Parma **5** (1954), 313–324.

Let $H_{n,m}(x, y; f(\xi, \eta))$ be the (unique) polynomial in x and y with the following properties: 1) $H_{n,m}$ is of degree $\leqq 2n-1$ in x, of degree $\leqq 2m-1$ in y. 2) $H_{n,m}$ assumes the same values as $f(x, y)$ at the points $(\cos(2j-1)\pi/2n, \cos(2k-1)\pi/2n)$ $(j=1, 2, \cdots, n;\ k=1, 2, \cdots, m)$. 3) The first partial derivatives of $H_{n,m}$ and its mixed second derivative vanish at these points. The author proves: if $f(x, y)\in C$ in Q: $-1\leqq x, y\leqq 1$, then $H_{n,m}(x, y; f)\to f$, uniformly in Q. If $f(x, y)$ satisfies a Lipschitz condition of order α, $\frac{1}{2}<\alpha\leqq 1$, then

$$H_{n,m}(x, y, f)-f(x, y)=O(n^{-\alpha}\log^{2-\alpha} n + m^{-\alpha}\log^{2-\alpha} m).$$

Let $\tau_{n,m}(x, y; f)$ be the trigonometric polynomial of order n in x, order m in y which agrees with $f(x, y)$ at the points $(2\pi j/(n+1); 2\pi k/(m+1))$ $(j=0, 1, \cdots, n;\ k=0, 1, \cdots, m)$ and has vanishing first partial derivatives and second mixed derivative at these points. Then $\tau_{n,m}(x, y)-f(x, y)\to 0$ for every $f(x, y)$ continuous in T: $0\leqq x, y\leqq 2\pi$. If f satisfies a Lipschitz condition of order α, $0<\alpha\leqq 1$, then $\tau_{n,m}-f=O(n^{-\alpha}\log n + m^{-\alpha}\log m)$. *W. J. H. Fuchs.*

Krylov, V. I. Convergence of algebraic interpolation in classes of differentiable functions. Dokl. Akad. Nauk SSSR (N.S.) **105** (1955), 214–217. (Russian)

Let $L_{n-1}(f, x)=\sum_{k=1}^{n} f(x_{nk})\omega_{nk}(x)$ be the Lagrange interpolating polynomial of $f(x)$ corresponding to the points x_{nk} $(k=1, \cdots, n)$ in $[a, b]$. Let $F_{nr}(t)$ denote the rth indefinite integral (properly normalized) of the function $F_{n0}(t)=\sum_{k=1}^{n} \omega_{nk}(x)E(t-x_{nk})$, where $E(t)=\frac{1}{2}(1+\text{sign } t)$. Writing the difference $f(x)-L_{n-1}(f, x)$ as a Stieltjes integral and integrating r times by parts, the author obtains necessary and sufficient conditions for convergence of $L_{n-1}(x)$ to $f(x)$ at a point x, or uniform convergence, when f belongs to one of the classes $C^{(r)}$, $A^{(r)}$, $V^{(r)}$ of functions with $f^{(r)}$ continuous, or absolutely continuous, or of bounded variation, respectively. For the class $A^{(r)}$, for example, the condition is $|F_{nr}(t)|\leqq N(x)$ for convergence at x, or $|F_{nr}(t)|\leqq N$ for uniform convergence in $[a, b]$. *G. G. Lorentz* (Detroit, Mich.).

Fekete, Michael. On the structure of polynomials of least deviation. Bull. Res. Council Israel. Sect. A. **5** (1955), 11–19.

Let S be a compact set in the z-plane, $f(z)$ continuous on S. A polynomial of degree $\leqq n$ is called a nearest polynomial to $f(z)$ on S, if there is no polynomial Q of degree $\leqq n$ satisfying

$$|Q(z)-f(z)|<|P(z)-f(z)|\quad (z\in S;\ P(z)-f(z)\neq 0).$$

If S has more than $n+2$ points and if the nearest polynomial P satisfies $P\neq f$ on S, then P can be characterized as a nearest polynomial of f on a suitable subset T of m points $(n+1\leqq m\leqq 2n+2)$ of S. Several relations between the values of P and f on T are given. These generalize the results of T. S. Motzkin and J. L. Walsh [Proc. Amer. Math. Soc. **4** (1953), 76–87; MR **15**, 701]. If $L(z)$ is the Lagrange interpolation polynomial to $f(z)$ on T and if the exact degree of $L(z)-P(z)$ is $n+k$ $(k\geqq 1)$, then T subtends an angle $\geqq \pi/k$ at at least n of the roots of $L-P$. *W. H. J. Fuchs* (Ithaca, N.Y.).

Szász, Pál. Remark on a theorem of Lipót Fejér. Eötvös L. Tud.-Egy. Kiadv. Term.-Tud. Kar Évk. **1952-53**, 15–18 (1954). (Hungarian)

Fejér proved the following theorem [Amer. Math. Monthly **41** (1934), 1–14]: Let $f(x)$ $(-1\leqq x\leqq 1)$ be continuous and $x_1, x_2, \cdots, x_n$ a normal point group. Denote

$$E_{2n-1}=\min \max_{-1\leqq x\leqq 1} |f(x)-P_{2n-1}(x)|,$$

$$E_{2n-1}^*=\min \max_{-1\leqq x\leqq 1} |f(x)-P_{2n-1}(x)|,$$

where, in E_{2n-1}, $P_{2n-1}(x)$ runs through all polynomials of degree $\leqq 2n-1$ and, in E_{2n-1}^*, $P_{2n-1}(x)$ runs through all polynomials of degree $2n-1$ for which $P_{2n-1}(x_i)=f(x_i)$ $(1\leqq i\leqq n)$. Then

$$E_{2n-1}^*\leqq 2E_n. \tag{1}$$

The author proves that if $x_i=\cos(2i-1)\pi/2n$ $(i=1, \cdots, n)$ (i.e. the x_i are the roots of the nth Tchebicheff polynomial $T_n(x)$, and are well-known to be normal) and $f(x)=2^{2n-1}x^{2n}$, then in (1) we have the sign of equality for all n (i.e. Fejér's result is best possible). In fact, $P_{2n-1}(x)=f(x)-2T_n^2(x)$ is the only polynomial of degree $2n-1$ for which $E_{2n-1}^*=2E_n$. *P. Erdös* (Haifa).

Fejér, Lipót. Elementary remarks concerning the fundamental polynomials of parabolic interpolation. Mat. Lapok **6** (1955), 293–308. (Hungarian. Russian and English summaries)

Let (x_k) be the n distinct abscissas of the Hermite interpolation, $\omega(x)=\prod(x-x_k)$; we denote by $h_k(x)$ the fundamental polynomials of the first kind, and by $H(x)$

the sum of the fundamental polynomials of the second kind. The following expansion is proved:

$$h_k(x)=1-\tfrac{1}{12}(5s_1^2+4s_2)(x-x_k)+\cdots,$$
$$s_1=\sum_{m=1}^{n-1}\frac{1}{x_k-\xi_m},\quad s_2=\sum_{m=1}^{n-1}\frac{1}{(x_k-\xi_m)^2},$$

where ξ_m are the zeros of $\omega'(x)$. Moreover, it is shown that the polynomial $H(x)$ of degree $2n-1$ has the zeros x_k and in addition $n-1$ zeros separating the x_k. Finally let $-1\leqq x_k\leqq 1$; under the assumption that the system is normal (i.e., the linear functions entering in the interpolation formula are of constant sign in $-1\leqq x\leqq 1$) the following inequality holds: $|H(x)-x|\leqq 1$ $(-1\leqq x\leqq 1)$.

G. Szegő (Stanford, Calif.).

Kakehashi, Tetsujiro. The decomposition of coefficients of power-series and the divergence of interpolation polynomials. Proc. Japan Acad. **31** (1955), 517–523.

Let the points $z_k^{(n)}$ $(k=1,\cdots,n;\ n=1,2,\cdots)$ lie in $|z|\leqq 1$ and be such that $(z-z_1^{(n)})(z-z_2^{(n)})\cdots(z-z_n^{(n)})z^{-n}$ converges uniformly on finite closed sets in $|z|>1$ to a function which is analytic and non-vanishing there. Let $f(z)$ be analytic throughout the interior of the circle C_ϱ: $|z|=\varrho>1$. The sequence of polynomials $P_n(z;f)$ of respective degrees n found by interpolation to f at $z_1^{(n+1)}$, $z_2^{(n+1)},\cdots,z_{n+1}^{(n+1)}$ diverges at every point exterior to C_ϱ. Moreover,

$$\limsup_{n\to\infty}|P_n(z;f)|^{1/n}=|z|/\varrho \text{ for } |z|>\varrho>1.$$

P. Davis (Washington, D.C.).

Rymarenko, B. A. On application of S. N. Bernšteĭn's method in the theory of monotonic polynomials. Dokl. Akad. Nauk SSSR (N.S.) **103** (1955), 373–375. (Russian)

Theorem. Let T be the class of polynomials $y(x)=\sum p_k x^k$ of degree $\leqq n$ with $p'(x)\geqq 0$ $(|x|\leqq 1)$ and satisfying $\sum_{k\leqq n}\alpha_k p_k=A$ and $y(-1)=0$. Then $\inf_{y\in T}[y(1)-y(-1)]$ is attained for a polynomial of the form

$$\int_{-1}^{x}(1-x)^\alpha(1+x)^\beta U^2(x)dx,$$

where α, β are 0 or 1 and U is a polynomial of degree $(n-1-\alpha-\beta)/2$ whose roots lie in $-1\leqq x\leqq 1$. A similar theorem is proved with T replaced by the class of polynomials increasing on the whole real axis and subject to two linear conditions on the coefficients.

W. H. J. Fuchs (Ithaca, N.Y.).

Ivanova, A. N. On convergence of a sequence of quadrature formulas of Gaussian type on infinite intervals. Dokl. Akad. Nauk SSSR (N.S.) **104** (1955), 169–172. (Russian)

Let the function $\varphi(x)\geqq 0$, $0<x<+\infty$, be such that the polynomials are dense in the space of continuous functions f with $\|f\|=\sup|f(x)/\varphi(x)|<+\infty$, and let $x\varphi'(x)/\varphi(x)\uparrow+\infty$, $\log\varphi(x)/\log^2 x\to\infty$ for $x\to+\infty$. Then the Gaussian interpolation formula of f with weight $1/\varphi(x)$ converges to $f(x)$ provided $f(x)$ is continuous and satisfies the condition $f(x)/\varphi(\lambda x)\to 0$ as $x\to\infty$ for some $0<\lambda<1$. A result of Videnskiĭ [same Dokl. **92** (1953), 217–220; MR **15**, 524] is used and some applications given. *G. G. Lorentz.*

Atkinson, F. V. On orthogonal polynomials with extrema at the ends of the orthogonality interval. Monatsh. Math. **59** (1955), 323–330.

The author is concerned with the following set $\{q_n(x)\}$ of orthogonal polynomials. The highest term of $q_n(x)$ is cx^{n+2}, $c>0$, for $n\geqq 1$, and $c>0$ for $n=0$. Moreover,

$$\int_{-1}^{+1}q_n(x)q_m(x)dx=\delta_{nm}$$

and $q_n'(\pm 1)=0$. It is not difficult to represent $q_n(x)$ in terms of Legendre polynomials. The following asymptotic formula is established:

$$q_n(\cos\theta)=(\tfrac{1}{2}\pi\sin\theta)^{-\frac{1}{2}}\cos[(n+5/2)\theta-\pi/4]+O[n^{-1}(\sin\theta)^{-5/4}],$$

where $n\sin\theta>A>0$. The zeros of $q_n(x)$ are all in $-1<x<1$, except two zeros $\pm y_n$, $y_n>1$. It is shown that $\lim_{n\to\infty}n^2(y_n-1)$ exists and coincides with the only positive root of the Bessel function

$$(1-6D+12D^2)I_0[(2\eta)^{\frac{1}{2}}]\quad(D=d/d\eta).$$

G. Szegő (Stanford, Calif.).

Tandori, Károly. On orthogonal series. Magyar Tud. Akad. Mat. Fiz. Oszt. Közl. **5** (1955), 477–479. (Hungarian)

Let $\{\varphi_n(x)\}$ be orthonormal in $[a,b]$ and $\sum a_k^2<\infty$. We assume that almost everywhere $(C,1)$-$\sum a_k\varphi_k(x)=f(x)$; then, almost everywhere,

$$\lim_{n\to\infty}\sigma_{2^n}^{(\alpha)}(x)=0$$

holds where $0<\alpha<1$ and

$$\sigma_n^{(\alpha)}(x)=[A_n^{(\alpha)}]^{-1}\sum_{\nu=0}^{n}A_{n-\nu}^{(\alpha-1)}[f(x)-s_\nu(x)]^2,$$
$$A_n^{(\alpha)}=\binom{n+\alpha}{n},\quad s_n(x)=\sum_{k=0}^{n}a_k\varphi_k(x).$$

G. Szegő (Stanford, Calif.).

Fuchssteiner, W. Über die Bildung der Koeffizienten bei der Entwicklung einer Funktion nach einem vorgeschriebenen Funktionensystem. Z. Angew. Math. Mech. **35** (1955), 184–190. (English, French and Russian summaries)

Let $\{m_n(x)\}$ be a sequence of functions to be used for expansion of arbitrary functions $f(x)$ on a given interval, and suppose the nth coefficient is given by a linear operator $L_n(f)=f^{\langle n\rangle}$. Thus, $f\sim\sum f^{\langle n\rangle}m_n(x)$. By use of Sylvester's determinant theorem, various recurrence relations are found. In particular, the coefficients are expressed in terms of the remainder $R_n(x)=f-\sum_0^{n-1}f^{\langle k\rangle}m_k$. By a process analogous to Gram-Schmidt orthogonalization $m_n(x)$ is specialized so that certain determinants simplify. This specialization applies to Taylor series, Fourier series, and Newton series. The author's methods for the latter yield an expansion involving Laguerre polynomials:

$$\frac{a^x}{\Gamma(x+1)}=\sum_0^\infty(-1)^k\frac{L_k(a)}{k!}\binom{x}{k}.$$

[The reviewer was unable to derive (7), but obtained the same with a factor $\Delta_{n-1}[m_{n-1}^{\langle n-1\rangle}]$ on the left. This leads to minor emendation of some later equations; in particular the product in (10) telescopes to two factors. The reviewer found the "normalization by means of $m_n(x)=\phi_n(x)/\phi_n^{\langle n\rangle}$" disturbing; there will exist such a ϕ only if m_n is already normalized. The matter is readily reworded, however. In connection with (17)–(19), the reviewer suggests that the equation preceding (5) yields $R_n^{\langle m\rangle}=f^{\langle m\rangle}-\sum f^{\langle k\rangle}m_k^{\langle m\rangle}$ when operated on by L. This gives an

explicit formula $B_{n_k}=-m_k^{(n)}$, without the assumption $M_n=m_n$.] *R. M. Redheffer* (Los Angeles, Calif.).

Eweida, M. T. On Turán's determinant for Legendre and Laguerre polynomials. Rev. Mat. Hisp.-Amer. (4) 15 (1955), 79–87.

A new proof is given for the inequality

$$\Delta_n(x)=[P_n(x)]^2-P_{n-1}(x)P_{n+1}(x)>0\ (-1<x<1),$$

and it is shown that $\Delta_n(x)$ assumes its maximum in $[-1, 1]$ at $x=0$. Other properties of $\Delta_n(x)$ are discussed as well as similar properties of the corresponding difference for Laguerre polynomials. *G. Szegö.*

Sponsler, George C. Two formulae involving generalized Legendre functions of integral and nonintegral index. J. Math. Phys. 34 (1955), 50–53.

The following two formulae are derived:

$$\text{i)}\quad (P_n(u_0))^{-1}=\sum_i \frac{2n_i+1}{(n_i-n)(n_i+n+1)}\left[-1\Big/\frac{\partial P_{n_i}(u_0)}{\partial n_i}\right],$$

where $|u_0|<1$, $P_n(u)$ is the nth Legendre polynomial, and the summation extends over all indices n_i (generally not integral) for which $P_{n_i}(u_0)=0$.

$$\text{ii)}\quad \int_{u_0}^1 P_\nu{}^m(u)P_n{}^m(u)du=\frac{(1-u_0{}^2)P_n{}^m(u_0)}{(\nu-n)(\nu+n+1)}\ \frac{\partial P_\nu{}^n(u_0)}{\partial u},$$

where $P_\nu{}^m(u_0)=0$, $P_n{}^m(u_0)\neq 0$, and $|u|$, $|u_0|<1$. The notation is that of Smythe, "Static and dynamic electricity" [2nd ed., McGraw-Hill, New York, 1950].
A. B. Novikoff (Baltimore, Md.).

Landau, H. G. Note on an inequality for the coefficients in Legendre polynomial expansions and its application to the theory of liquid phase transitions. Bull. Math. Biophys. 17 (1955), 41–44.

The author establishes the following theorem: Let

$$f(x)=\sum_0^\infty A_nP_n(x)\ (-1\leq x\leq 1),$$

where $P_n(x)=n$th Legendre polynomial. If $f^{(m)}(x)\geq 0$, then $A_n/(2n+1)$ is non-increasing with n. An application is indicated suggesting as a consequence the physical realism of certain biophysical models.
A. B. Novikoff (Baltimore, Md.).

See also: Džrbašyan and Tamadyan, p. 598.

Trigonometric Series and Integrals

Žak, I. E. On some applications of theorems of S. N. Bernšteĭn and I. I. Privalov. Soobšč. Akad. Nauk Gruzin. SSR 16 (1955), 185–190. (Russian)

The Bernstein theorem referred to states; If $t_n(x)$ is any trigonometric polynomial and $|t_n(x)|\leq C$ for all x, then $|t_n'(x)|\leq Cn$. The Privalov theorem referred to states: If $|t_n(x)|\leq C$ for all $a\leq x\leq b$, $[a, b]\subset(-\pi,\pi)$, then $|t_n'(x)|\leq K_\varepsilon Cn$ for all $[a+\varepsilon\leq x\leq b-\varepsilon]$, where $K_\varepsilon\to+\infty$ as $\varepsilon\to 0+$. Both statements have extensions to double trigonometric polynomials. Let T_1 denote any double trigonometric series in the variables x, y, and T_2, T_3, T_4 its three conjugate series. If (a) three of the four series T_r are uniformly convergent in the xy-plane, then the fourth one is convergent almost everywhere. If the four series T_r are the Fourier series of continuous functions then (a) implies that also the fourth series is uniformly convergent. If $|T_{mn}{}^{(r)}(x,y)-f_r(x,y)|\leq C(m^{-\alpha}+n^{-\alpha})$ for all $(x,y)\in[a,b;c,d]$, $1>\alpha>0$, for some r, where $T_{mn}{}^{(r)}$ denotes the mn-partial sum of the series T_r, then $f_r(x,y)\in$ Lip α in every rectangle $[a+\varepsilon, b-\varepsilon; c+\varepsilon, d-\varepsilon]$. *L. Cesari* (Lafayette, Ind.).

Szász, Pál. Über das Restglied der Sinusreihe

$$\sum_{n=1}^\infty \frac{\sin nx}{n}.$$

Mat. Lapok 6 (1955), 130–137. (Hungarian. Russian and German summaries)

Based on a formula of L. Fejér the author proves the following identity:

$$\varrho_n(x)=\frac{\pi-x}{2}-\sum_{\nu=1}^n\frac{\sin\nu x}{\nu}=$$
$$\frac{1}{2(n+1)}\left\{\frac{\cos\frac12(2n+1)x}{\sin\frac12 x}-\frac12\int_x^\pi\frac{\cos(n+1)t}{\sin^2\frac12 t}dt\right\},$$

from which in the same interval $0<x<2\pi$,

$$(2n+1)\varrho_n(x)=\frac{\cos\frac12(2n+1)x}{\sin\frac12 x}+\eta_n(x),\quad \eta_n(x)\to 0 \text{ as } n\to\infty,$$

follows. The convergence is uniform in every interval $(\delta, 2\pi-\delta)$ where $0<\delta<\pi$. *G. Szegö.*

Czipszer, János, and Rényi, Alfréd. On the completeness of certain trigonometric systems. Magyar Tud. Akad. Mat. Fiz. Oszt. Közl. 5 (1955), 391–410. (Hungarian)

The authors give a complete account of the complex values of τ for which the sets

$$(1)\ \{\cos(k+\tau)x\},\ (2)\ \{1, \cos(k+\tau)x\},\ (3)\ \{\sin(k+\tau)x\}\quad (k=0, 1, 2, \cdots)$$

are complete in $[0, \pi]$. The completeness is meant in the space L^p, $1\leq p<\infty$, as well as in C and C^0, respectively. [C^0 is the space of all functions in C vanishing at $x=0$.] For instance, in the case (1) the necessary and sufficient condition is:

$$\begin{aligned}&\text{Re }\tau\leq\tfrac12+1/(2p) \text{ in } L^p\ (p>1),\\ &\text{Re }\tau<1 \text{ in } L,\\ &\text{Re }\tau\leq\tfrac12,\ \tau\neq\tfrac12, -\tfrac12, -\tfrac32, \cdots \text{ in } C.\end{aligned}$$

The set (1) is never complete in C^0. In the proof of the non-completeness the formula

$$\frac{1}{\pi}\int_0^\pi(2\cos\tfrac12 x)^{a-1}\cos\tfrac12 bx\,dx=\frac{\Gamma(a)}{\Gamma(\frac12(a+b+1))\Gamma(\frac12(a-b+1))};\ \text{Re } a>0,$$

and similar formulas are used. Two other trigonometric sets are also considered, refining a result of K. Šaĭdukov [Uspehi Mat. Nauk (N.S.) 8 (1953), no. 6(58), 143–153; MR 16, 241]. *G. Szegö* (Stanford, Calif.).

Izumi, Shin-ichi. Some trigonometrical series. XVI. Proc. Japan Acad. 31 (1955), 511–512.

Let $s_n(x)$ be the nth partial sum of the Fourier series of the function $f(x)$, of power series type. If

$$\int_0^{2\pi}|f(x+t)-f(x)|^p dx\leq(At^{1/\alpha})/(\log 1/t)^{(1+\varepsilon)/\alpha},$$

where $1<\alpha\leq p$ and $\varepsilon>0$, then $\sum|s_n(x)-f(x)|^\alpha$ converges almost everywhere. *P. Civin* (Eugene, Ore.).

Kinukawa, Masakiti. On the convergence character of Fourier series. Proc. Japan Acad. 31 (1955), 513–516.

Let $s_n(x)$ be the nth partial sum of the Fourier series of $f(x)$. The uniform convergence of

$$\sum |s_n(x)-f(x)|^k/[n^\delta(\log n)^\gamma],$$

or special cases thereof, is shown under the following hypotheses. (1) $f(x) \in \mathrm{Lip}\,\alpha$ $(0<\alpha<\frac{1}{2})$, $\delta=1-k\alpha$, $\gamma>1$, $k\alpha<1$ and $k>0$. (2) $\delta=0$: $f(x) \in \mathrm{Lip}\,\alpha$, $k\alpha=1$, $\gamma>(1-\alpha)/\alpha$ and $k\geqq 2$. (3) $\gamma=0$: $|f(x+t)-f(x)|=O\{|t|^\alpha/(\log 1/|t|)^\sigma\}$ uniformly, $0<\alpha<\frac{1}{2}$, $\delta=1-k\alpha$, $k\alpha<1$, $k>0$ and $\sigma>1/k$. (4) $\gamma=0$ and $\delta=0$: $|f(x+t)-f(x)|=O\{|t|^\alpha/(\log(1/|t|))^\sigma\}$ uniformly, $k\alpha=1$, $k\geqq 2$ and $1+k(\sigma-1)>0$. *P. Civin.*

Keogh, F. R. Summability of a class of Fourier-Stieltjes series. J. London Math. Soc. 31 (1956), 64–67.

Let $\frac{1}{2}a_0+\sum_1^\infty (a_n \sin n\theta+b_n \cos n\theta)$ be the Fourier-Stieltjes series of the function $F(t)$ which is assumed of bounded variation in $(0, 2\pi)$. Suppose that $F(t)$ is increasing in the neighborhood of $t=\theta_0$. Then the author proves that at θ_0 Abel summability and (C, α) summability, $\alpha>0$, are equivalent. A necessary condition and a sufficient condition for summability under these circumstances are also presented. The results are generalizations of similar theorems for Fourier series obtained by Hardy and Littlewood. *J. Blackman* (Syracuse, N.Y.).

Sinvhal, S. D. A note on a theorem of F. C. Hsiang. Ann. Soc. Sci. Bruxelles. Sér. I. 69 (1955), 79–84.

It is claimed that a demonstration by Hsiang [Duke Math. J. 13 (1946), 43–50; MR 7, 293] is incorrect. The allegation is based, however, on an improper usage of the theorem relating the inferior and superior limits of a sequence with those of its transform by a positive Toeplitz matrix. *P. Civin* (Eugene, Ore.).

Chow, Hung Ching. Criteria for the strong summability of the derived Fourier series and its conjugate series. J. London Math. Soc. 31 (1956), 57–64.

Let $f(t)$ be a Lebesgue integrable function over the interval $(0, 2\pi)$ and with period 2π, and its Fourier series be

$$\tfrac{1}{2}a_0+\sum_{n=1}^\infty (a_n \cos nt+b_n \sin nt).$$

Then the derived Fourier series becomes

$$(*)\qquad -\sum_{n=1}^\infty n(a_n \sin nt-b_n \cos nt)=\sum_{n=1}^\infty nB_n(t).$$

The author proves that, if $\sum_{n=1}^m n|B_n(x)|=o(m)$ and the function $\chi(t)=(f(x+t)-f(x-t))/2t$ is of bounded variation in a neighborhood of $t=0$, then the series (*) is strongly summable at $t=x$, that is

$$\sum_{n=1}^m |\sigma_n(x)-\chi(+0)|=o(m),$$

where $\sigma_n(x)$ is the nth partial sum of the series (*). This theorem generalizes a theorem of B. N. Prasad and U. N. Singh [Math. Z. 56 (1952), 280–288; 57 (1953), 481–482; MR 14, 370; cf. the reviewer and M. Kinukawa, Proc. Japan Acad. 31 (1955), 107–110; MR 17, 32]. The author proves also a similar theorem concerning the conjugate series of (*). *S. Izumi* (Sapporo).

See also: Korevaar, p. 594; Łojasiewicz, Wloka and Zieleźny, p. 594; Sandor, p. 613; Edwards, p. 645; Herz, p. 645.

Integral Transforms, Operational Calculus

Ganelius, Tord. Un théorème taubérien pour la transformation de Laplace. C. R. Acad. Sci. Paris 242 (1956), 719–721.

Recently several authors have obtained estimates of the remainder in Tauberian theorems for power series, Dirichlet series and Laplace transforms. Estimates based on behavior of the transforms on the real axis were obtained by A. G. Postnikov [Dokl. Akad. Nauk SSSR (N.S.) 77 (1951), 193–196; MR 12, 820], G. Freud [Acta Math. Acad. Sci. Hungar. 2 (1951), 299–308; 3 (1952), 299–307; 5 (1954) 275–289; MR 14, 361, 958, 17, 260; Acta Sci. Math. Szeged 16 (1955), 12–28; MR 17, 30] and the reviewer [Duke Math. J. 18 (1951), 723–734; Nederl. Akad. Wetensch. Proc. Ser. A. 56 (1953), 281–293; 57 (1954), 36–45, 46–56, 152–160, 432–443, 444–455; MR 13, 227; 15, 119, 698, 950; 16, 239]. The most notable advance in these estimates came when Freud and the reviewer independently replaced the customary uniform approximation in the Karamata-Wielandt proof by approximation in the sense of L_1. The present author uses similar methods. However, by introducing Schmidt's Tauberian condition into the Tauberian theorems with remainder he is able to announce more elegant results. The following special case is typical. As $\omega\to\infty$ let $Q(\omega)/\omega\uparrow\infty$, let

$$\int_0^\infty e^{-\lambda/\omega}d\alpha(\lambda)=O\{\omega^{\rho+1}\exp(-2Q(\omega)/\omega)\}$$

and let $\alpha(\Omega)-\alpha(\omega)\leqq C\omega^\rho q(\omega)$ for $\omega\leqq\Omega\leqq\omega+q(\omega)$, where q is the inverse function of Q. Then $\alpha(\omega)=\alpha(0)+O\{\omega^\rho q(\omega)\}$ as $\omega\to\infty$. The special case $Q(\omega)=\omega^r$ $(1<r\leqq 2)$ of this result was considered earlier by V. Vučković [Srpska Akad. Nauka. Zb. Rad. 35. Mat. Inst. 3 (1953), 255–288; MR 15, 869]. Vučković derived his result from the known estimates of the remainder which are based on the behavior of transforms in the complex plane. *J. Korevaar* (Madison, Wis.).

Akutowicz, Edwin J. The uniqueness of Laplace integrals. Duke Math. J. 23 (1956), 165–174.

Let $\Phi(z)=\int_0^\infty e^{-zt}\phi(t)dt$. The author presents six theorems which treat various conditions on $\phi(t)$ and the zeros of $\Phi(z)\equiv 0$. These stem either from a factorization theorem of Hille and Tamarkin [Fund. Math. 25 (1935), 329–352] or from a modification of Carleman's formula. The idea behind all the theorems is that if $\Phi(z)$ is a Laplace transform then growth conditions on $\phi(t)$ (e.g., $\phi \in L^p(0, \infty)$) imply growth conditions on the analytic function $\Phi(z)$ which when coupled with a sufficient number of zeros guarantee that Φ must vanish identically. A typical example of the type of theorem treated is theorem 6: If ϕ is bounded and its Laplace transform $\Phi(z)$ vanishes at real points x_n, $x_n\downarrow 0$ $(n\to\infty)$, such that

$$\lim_{L\to\infty}\frac{1}{L}\sum_{x_n>\exp(-L)}(x_n-\exp(-L))=+\infty,$$

then $\Phi\equiv 0$. *J. Blackman* (Syracuse, N.Y.).

Rooney, P. G. On an inversion formula for the Laplace transformation. II. Canad. J. Math. 8 (1956), 49–52.

Let $f(s)=\int_0^\infty e^{-st}\phi(t)dt$. In a previous paper [same J. 7 (1955), 101–115; MR 16, 584]. The author considered the inversion of this equation using the operator ${}^\nu L_{k,t}[f(s)]$ (originally introduced by Erdélyi) for $\nu>-1$. The present

paper extends some of the results to the case $\nu<-1$, $\nu\neq$an integer. *J. Blackman* (Syracuse, N.Y.).

Stanković, Bogoljub. Inversion et invariants de la transformation généralisée de Hankel. C. R. Acad. Sci. Paris **241** (1955), 1905–1907.

The generalized Hankel transformation is

$$G(x)=\int_0^\infty \Phi(\mu+1, \nu; -x^\nu y)y^\mu g(y)dy,$$

where $\mu>-1$, $\nu>0$, and

$$\Phi(\mu, \nu; z)=\sum_{k=0}^\infty \frac{z^k}{k!\Gamma(\mu+\nu k)}.$$

This transformation reduces to the Hankel transformation when $\nu=1$. For the general transformation, the note contains an inversion formula (which is an integral transform of the same form), and necessary and sufficient conditions to be satisfied by the Laplace transform of $t^\mu g(t)$ if $g(x)$ is self-reciprocal. [See also R. P. Agarwal, Ann. Soc. Sci. Bruxelles. Sér. I. **64** (1950), 164–168; Bull. Calcutta Math. Soc. **45** (1953), 69–73; MR **12**, 605; **15**, 524.] *A. Erdélyi* (Pasadena, Calif.).

See also: Korenblyum, p. 605; Edwards, p. 645; Rooney, p. 646; Ahlfors, p. 657.

Special Functions

Uhler, Horace S. Nine exact factorials between 449! **and** 751! Scripta Math. **21** (1955), 138–145.

The author gives $n!$ for $n=500$, 550, 563, 600, 603, 650, 652, 700 and 750 the terminal zeros being omitted to save space. Thus 750! would end in 174 zero digits. Various methods used in checking these results are discussed, including K. Goldberg's table of Wilson's quotients

$$w_p=[(p-1)!+1]/p$$

[J. London Math. Soc. **28** (1953), 252–256; MR **14**, 1062]. The reviewer is puzzled by the author's remark that $563!+563$ is divisible by 563^2. Actually it should be divisible by 563^3. *D. H. Lehmer* (Berkeley, Calif.).

Chak, A. M. A class of polynomials and a generalization of Stirling numbers. Duke Math. J. **23** (1956), 45–55.

The author considers some polynomials defined by differentiation formulas or generating functions; and also certain numbers occurring in formulas relating to, or connecting different classes of, his polynomials. All of the polynomials are generalizations of known polynomials, but no motivation is presented for the generalization, and all the results are of the manipulative kind. Sample:

$$G_{n,k}^{(\alpha)}(x)=x^{-\alpha-kn+n}e^x\left(x^k\frac{d}{dx}\right)^n[x^\alpha e^{-x}]$$

is a generalization of Laguerre polynomials and is generated by

$$(1-u)^{\alpha/(k-1)}\exp[x-x(1-u)^{-1/(k-1)}]=\sum_{n=0}^\infty \frac{u^n}{n!}(k-1)^n G_{n,k}^{(\alpha)}(x).$$

A. Erdélyi (Pasadena, Calif.).

Nørlund, N. E. Hypergeometric functions. Acta Math. **94** (1955), 289–349.

This is one of those rare papers in which sound mathematics goes hand in hand with excellent exposition and style; and the reader is both instructed and delighted. It is likely to become the standard memoir on the generalized hypergeometric series ${}_nF_{n-1}$.

The author studies the hypergeometric differential equation $Q(\delta)y-zR(\delta)y=0$, where $\delta=zd/dz$, $Q(x)=(x-\gamma_1)\times\cdots(x-\gamma_n)$, and $R(x)=(x+a_1)\cdots(x+a_n)$. The equation has three singularities, at $z=0$, 1, ∞. At $z=0$, the exponents are γ_s, and the corresponding solutions (with leading coefficient 1), y_s, can be expressed in terms of ${}_nF_{n-1}$; likewise the solutions $\bar{y}_s$ belonging to the exponents a_s at $z=\infty$ ($s=1, \cdots, n$). The former series converge for $|z|<1$, the latter for $|z|>1$; if $\beta_n=n-1-(a_1+\gamma_1)-\cdots-(a_n+\gamma_n)$ satisfies $\operatorname{Re}\beta_n>0$ both series converge absolutely and if $0\geq\operatorname{Re}\beta_n>-1$ they converge conditionally, for $|z|=1$, $z\neq1$. If some of the γ_s (or a_s) differ by integers, logarithmic solutions arise. These are given, as are several transformation formulas. The third singularity, $z=1$, presents greater difficulties. One solution belongs to the exponent β_n there, and for this solution, $\xi_n(z)$, several expansions are given, with interesting relations satisfied by the coefficients. The other solutions belong to the exponents 0, 1, $\cdots$, $n-2$. Here $\xi_n(z)$ can be represented by an integral involving $\xi_{n-1}(z)$, the integral $\int_0^1 z^s\xi_n(z)dz$ can be expressed as a combination of gamma functions, and this leads to an inverse factorial series generating the coefficients of $\xi_n(z)$, and also to a Mellin-Barnes integral representing $\xi_n(z)$.

The $y_s(z)$ and $\bar{y}_s(z)$ can also be represented by Mellin-Barnes integrals, and y_s can be expressed as a linear combination of $\bar{y}_1, \cdots, \bar{y}_n$ or vice versa. The modified solutions needed in the case of integer exponent-differences are also represented by integrals.

Those solutions belonging to the exponents 0, 1, $\cdots$, $n-2$ at $z=1$ can be represented in terms of the functions $y_s^*(z)-y_h^*(z)$, where y_s^* is a constant multiple of y_s. Except in the case of integer β_n, $n-1$ of these differences are linearly independent and together with $\xi_n(z)$ form a fundamental system. The new solutions are represented by Mellin-Barnes integrals and, most elegantly, by infinite series of hypergeometric polynomials. If β_n is an integer, solutions with logarithmic singularities occur, and these are also investigated.

This is only a brief outline of the paper. In a brief review it is impossible to describe the numerous detailed results or to convey an impression of the unhurried presentation. *A. Erdélyi* (Pasadena, Calif.).

Robin, Louis. Dérivée de la fonction associée de Legendre de première espèce, par rapport à son degré. C. R. Acad. Sci. Paris **242** (1956), 57–59.

The author gives a series expansion of $\partial P_n{}^m(x)/\partial n$ for non-integer n, computes the limiting form for integer n, and obtains a recurrence formula involving $\partial P_n/\partial n$ and P_n. *A. Erdélyi* (Pasadena, Calif.).

See also: Jeffreys, p. 604; Eweida, p. 608; Schuler und Gebelein, p. 670; Rahman, p. 671; Sternberg, Shipman and Kaufman, p. 671; Binnie and Miller, p. 672; Beer, Chase and Choquard, p. 672.

Ordinary Differential Equations

Hukuhara, Masuo. Sur l'existence des solutions des équations différentielles ordinaires. Proc. Japan Acad. **31** (1955), 391–394.

A result of Nagumo's [Proc. Phys.-Math. Soc. Japan

(3) 24 (1942), 551–559; MR 7, 381] for the system $Y'=F(x, Y)$, with F continuous, is here extended to the case in which F is continuous in Y and if $Y(x)$ is continuous then $F(x, Y(x))$ is measurable. *F. A. Ficken.*

Alexiewicz, A.; and Orlicz, W. On a theorem of C. Carathéodory. Ann. Polon. Math. 1 (1955), 414–417.

For $a\leqq t\leqq b$ and $\alpha\leqq u\leqq\beta$ let $\varphi(t, u)$ be continuous in u and, for u in a set dense in $[\alpha, \beta]$, measurable in t, and suppose that $|\varphi(t, u)|\leqq s(t)$ with s integrable. If $a<\tau<b$ and $\alpha<\eta<\beta$, then there exist numbers p and q such that $p<\tau<q$ and an absolutely continuous function $y(t)$ defined for $p<t<q$ such that $y(\tau)=\eta$ and $y'(t)=\varphi(t, y(t))$ for almost all t. This slightly strengthened form of Carathéodory's theorem [Vorlesungen über reelle Funktionen, 2nd ed., Teubner, Leipzig-Berlin, 1927, p. 672] is deduced from the fact that, under the conditions here imposed, there exist continuous functions $\varphi_n(t, u)$ such that $|\varphi_n(t, u)|\leqq s(t)$ and, for almost all t in $[a, b]$,

$$\lim_{n\to\infty} \max_{\alpha\leqq u\leqq\beta} |\varphi_n(t, u)-\varphi(t, u)|=0.$$

This approximation theorem follows, in turn, from consideration of $\varphi(t, u)$ as a vector-valued function on $[a, b]$ into the space of functions continuous on $[\alpha, \beta]$, and on a criterion for measurability of vector-valued functions given by Pettis [Duke Math. J. 5 (1939), 254–269].

F. A. Ficken (Knoxville, Tenn.).

Pogorzelski, W. Problème aux limites de Poincaré généralisé. Ann. Polon. Math. 2 (1955), 257–270 (1956).

Detailed exposition of results announced in Bull. Acad. Polon. Sci. Cl. III. 3 (1955), 195–198; MR 17, 365.

Corduneanu, C. Théorèmes d'existence sur l'axe réel des solutions des équations différentielles nonlinéaires du second ordre. Acad. R. P. Romîne. Bul. Şti. Secţ. Şti. Mat. Fiz. 7 (1955), 645–651. (Romanian. Russian and French summaries)

Let $f(x, y, z)$ be continuous in x and have $\partial f/\partial y$, $\partial f/\partial z$ satisfying $0<m\leqq\partial f/\partial y\leqq M$, $|\partial f/\partial z|\leqq A$; let also $|f(x, 0, 0)|\leqq L$, $AM^{\frac{1}{2}}<m$. Then $y''=f(x, y, y')$ has a unique solution such that y, y' are bounded on $-\infty<x<\infty$. The proof is by successive approximation, defining y_n as the bounded solution of $y_n''-My_n=f(x, y_{n-1}, y_{n-1}')-My_{n-1}$. The special case $y''=f(x, y)$ is considered separately. In a recent paper of the author [Com. Acad. R. P. Romîne 5 (1955), 793–797; MR 17, 39] the existence of a bounded solution was a hypothesis. *F. V. Atkinson* (Canberra).

Bass, Robert W. On the regular solutions at a point of singularity of a system of non-linear differential equations. Amer. J. Math. 77 (1955), 734–742.

The author constructs regular solutions of a system of non-linear ordinary differential equations near a certain type of irregular singular point. Let $f_i(z, w_1, \cdots, w_n)$ be n analytic functions of $n+1$ complex variables near $(z, w_1, \cdots, w_n)=(0, 0, \cdots, 0)$ and suppose $f_i(z, 0, 0, \cdots, 0)\equiv0$. Theorem. The system $z^{s_i}w_i'(z)=f_i(z, w_1, \cdots, w_n)$ $(i=1, 2, \cdots, n)$ and $s=s_1+s_2+\cdots+s_n<n$, has at least an $(n-s)$-parameter algebroid family of solutions regular at $z=0$. The proof of this theorem, and of a slightly more general statement, use the method of comparison of coefficients. The resulting infinite set of non-linear equations is solved by means of Wintner's fixed-point theorem in an appropriate Hilbert space. *L. Markus.*

Krzywicka, E. Sur les solutions de l'équation différentielle $x^{(n)}+A(t)x=0$ qui satisfont à des conditions données dans plusieurs points. Bull. Acad. Polon. Sci. Cl. III. 3 (1955), 521–522.

Consider real solutions of $x^{(n)}+A(t)x=0$ where $A(t)\neq0$ is real and continuous on the real line. At $r\leqq n$ different points $\alpha_1<\alpha_2<\cdots<\alpha_r$ we examine the derivatives $x^{(j_{\mu\nu}-1)}(\alpha_\mu)=C_{\mu\nu}$, for a total of n boundary conditions. The author announces the following theorem. If the points $\alpha_1, \cdots, \alpha_r$ and the order q_μ of the highest derivative evaluated at each α_μ are fixed, then the existence of a unique solution satisfying the above auxiliary conditions is independent of the values $C_{\mu\nu}$ and of the natural numbers $j_{\mu\nu}$. Also the author remarks on conditions on $(q_1, q_2, \cdots, q_r)$ which assure the existence of a unique solution for all sequences $\alpha_1<\alpha_2<\cdots<\alpha_r$ of real numbers.

L. Markus (Princeton, N.J.).

Saltykow, N. Ordre d'un système d'équations différentielles ordinaires de la forme générale. Glas Srpske Akad. Nauka 206. Od. Prirod.-Mat. Nauka (N.S.) 5 (1953), 17–29. (Serbo-Croatian. French summary)

L'auteur détermine l'ordre du système de n équations différentielles ordinaires, compatibles et irréductibles par différentiation, qui ne sont pas résolubles par rapport aux dérivées des ordres supérieurs des fonctions inconnues y_i. On suppose que l'ordre supérieur de la dérivée de y_i se trouve précisément dans la i-ème équation du système consideré. *M. Zlámal* (Brno).

Ważewski, T. Sur les intégrales de branchement des systèmes des équations différentielles ordinaires. Ann. Polon. Math. 1 (1955), 338–345.

Differential systems are considered: (1) $dx/dt=F(t, X)$, $X=(x_1, \cdots, x_n)$, where F is a continuous real vector function of t and X in a $(n+1)$-dimensional open set W of the real (t, X)-space, and a uniqueness theorem holds. For every integral $J=J(P)$: $X=\phi(t)$, $\alpha<t<\beta$, passing through a point $P\in W$ the maximal interval of existence $M(P)=[r(P)<t<s(P)]$ is considered, where $r(P)$ may be $-\infty$ and $s(P)$ may be $+\infty$. The function $r(P)[s(P)]$ is upper [lower] semicontinuous. A sequence $[J_k]$ of integrals of (1) is said to be condensed on a point $Q\in W$ if for each k there is a point $P_k\in J_k$ with $P_k\to Q$ as $k\to+\infty$. The same sequence $[J_k]$ is said to be condensed on an integral J if it is condensed on each point $Q\in J$. Two distinct integrals J', J'' are said to be associated if there is a sequence $[J_k]$ which is condensed on both J', J''. An integral J is said to be a branch integral if there is at least another distinct integral J' associated to J. It is easily proved that any two associated integrals have disjoint maximal intervals of existence. In the present paper the author proves that the set $K\subset W$ of all points $P\in W$ which are on some branch integrals of (1) is of first category of Baire. The same concept of branch-integral was considered by the author in a paper concerning first-order partial differential equations [Mathematica, Cluj 8 (1934), 103–116] and more recently by J. Szarski [Ann. Soc. Polon. Math. 19 (1946), 106–132; MR 9, 186] and Z. Szmydtówna [same Ann. 23 (1950), 167–182; MR 13, 236].

L. Cesari (Lafayette, Ind.).

Hartman, Philip; and Wintner, Aurel. On disconjugate differential systems. Canad. J. Math. 8 (1956), 72–81.

The authors consider equations of the form

$$(1)\qquad (G(t)x')'+F(t)x=0$$

and

(2) $$x''+G(t)x'+F(t)x=0,$$

where F, G are n-by-n matrices whose elements are real-valued functions of t ($a\leqq t\leqq b$). By a solution of (1) (or (2)) is meant a vector $x(t)$ with real components which satisfies the equation and which does not vanish identically. An equation (1) or (2) is called disconjugate on the interval $a\leqq t\leqq b$ if there is no solution $x(t)$ with $x(t_1)=x(t_2)=0$ for $a\leqq t_1\leqq t_2\leqq b$. The authors establish sufficient conditions for (1) or (2) to be disconjugate on an interval. Denote by A^* the transpose of a matrix A, let $A^0=\frac{1}{2}(A+A^*)$. By $A\leqq B$ ($A<B$) is meant that the quadratic form belonging to B^0-A^0 is non-negative definite (positive definite). The authors prove that if the matrices P_j, Q_j ($j=1, 2$), in the system (3_j) $(P_j(t)x')'+Q_j(t)x=0$ satisfy the inequalities $0<P_1\leqq P_2$, $Q_1\geqq Q_2$ on $a\leqq t\leqq b$ and if (3_1) is self-adjoint, then (3_2) is disconjugate on $a\leqq t\leqq b$ whenever (3_1) is. The proof uses the corresponding theorem, due to M. Morse, for the case that (3_2) is self-adjoint.

The authors prove that (2) is disconjugate on $a\leqq t\leqq b$ whenever there is a matrix $K(t)$ satisfying $F-K'\leqq -(\frac{1}{2}G-K^0)(\frac{1}{2}G^*-K^0)$ on this interval. Hence (4) $F\leqq -\frac{1}{4}GG^*$ or (5) $F\leqq G'-\frac{1}{4}G^*G$ are each sufficient. The constant $\frac{1}{4}$ is best possible. The authors show also that (2) is disconjugate on an interval whenever its adjoint is and conversely.

The preceding results are used to obtain an existence theorem for solutions of (2) on the interval $0\leqq t<\infty$. Specifically, if (4) is satisfied for large t, then a system (2) of order $2n$ admits exactly n linearly independent solution vectors $x_i(t)$ such that all $|x_i(t)|$ tend to finite limits as $t\to\infty$. The proof is based on the fact that if $x(t)$ is a solution of (2), $|x|^2$ is a convex function of t in any interval in which (4) holds.

R. Finn (Los Angeles, Calif.).

Švec, Marko. Sur les dispersions des intégrales de l'équation $y^{(4)}+Q(x)y=0$. Czechoslovak Math. J. **5 (80)** (1955), 29–60. (Russian summary)

The subject of this paper is the location of the zeros of certain solutions of the differential equation (A) $y^4+Q(x)y=0$ when $Q(x)$ is positive and continuous for $-\infty<x<\infty$ and all solutions of (A) are oscillatory. Let M_{ik}, $i<k$, $k=0, 1, 2, 3$, denote the set of those solutions of (A) whose ith and kth derivatives vanish at a preassigned point x_1. The elements of M_{ik} satisfy a certain linear second-order differential equation. This fact enables the author to derive separation theorems for the zeros of these functions. Let x be arbitrary and consider an element y of M_{ik} that vanishes at x. The abscissa $\varphi_\nu(x)$ ($\nu=\pm 1, \pm 2, \cdots$) of the νth zero following (or preceding) x is called the νth central dispersion of y, in accordance with the terminology of O. Borůvka [same J. 3(78) (1953), 199–255; MR **15**, 706]. With the help of Borůvka's theory numerous properties of $\varphi_\nu(x)$ are derived of which the following may be mentioned. The central dispersions are increasing functions of x; they possess five continuous derivatives except possibly if $\varphi_\nu(x)=x_1$; they satisfy a nonlinear third-order differential equation. In the last part of the paper the author considers the linear transformations in M_{ik} regarded as a two-dimensional linear vector space. He is particularly interested in the effect of these transformations on the zeros of the elements of M_{ik}. The results are, however, too lengthy to be stated in tihs review.

W. Wasow (Los Angeles, Calif.).

Laitoch, Miroslav. Eine Erweiterung der Methode Floquets zur Darstellung des Fundamentalsystems von Lösungen der Differentialgleichung zweiter Ordnung $y''=Q(x)y$. Czechoslovak Math. J. **5(80)** (1955), 164–174. (Russian. German summary)

If $Q(x+1)=Q(x)$, then it is known from Floquet's theory that the differential equation $y''=Q(x)y$ possesses, in general, a fundamental system of the form $\exp\{x \log s_j\}\cdot \pi_j(x)$ ($j=1, 2$), where $\pi_j(x+1)=\pi_j(x)$. The author shows that this theorem admits the following generalization to not necessarily periodic continuous functions $Q(x)$. Let $\varphi(x)$ be a solution of the differential equation $\varphi'^{\frac{1}{2}}(1/\varphi'^{\frac{1}{2}})''+\varphi'^2Q(\varphi)=Q(x)$ and let $F[\varphi(x)]-F(x)=1$. Then the differential equation $y''=Q(x)y$ possesses, in general, a fundamental system of the form

$$U_j(x)=\exp\{F(x)\log s_j\}\pi_j(x)/\sqrt{F'(x)}\ (j=1, 2),$$

where $\pi_j(\varphi(x))=\pi_j(x)$. As in Floquet's theory the s_j are the roots of a certain quadratic equation. The results stated assume that $s_1\neq s_2$. They must be replaced by a more complicated statement when $s_1=s_2$. Applications to functions $Q(x)$ for which $k^2Q(kx)=Q(x)$ or $(1-mx)^{-4}\cdot Q(x/(1-mx))=Q(x)$ are included.

W. Wasow.

★ **Uno, Toshio. On some systematic method for finding limit cycles.** Proceedings of the First Japan National Congress for Applied Mechanics, 1951, pp. 513–516. Science Council of Japan, Tokyo, 1952.

The author considers the differential equation

$$\frac{dy}{dx}=\frac{AY+BX}{CY+DX},$$

where $X(x, y)$, $Y(x, y)$ are single-valued analytic functions which are real for real values of the arguments, and where A, B, C, D are real parameters such that $AD-BC=1$. Let E_1 and E_2 denote two equations, corresponding to parameter values (A_1, B_1, C_1, D_1), (A_2, B_2, C_2, D_2), respectively. E_1 is said to precede E_2 if, at every nonsingular point (x, y), the tangent to the integral curve of E_2 makes a positive angle with the tangent to the integral curve of E_1. A one-parameter subfamily of equations which can be ordered in this manner is said to be a locally ordered family. The notion of a locally ordered family of equations is applied in connection with certain questions concerning limit cycles. The principal results are the following, which are stated with brief indications of the proofs. (1) If, in a locally ordered family, one equation has a stable or unstable limit cycle, any succeeding equation has a corresponding limit cycle. (2) If, in a locally ordered family, one equation has a double limit cycle, formed by the confluence of a stable and an unstable limit cycle, any succeeding equation has 0 or 2 corresponding limit cycles.

L. A. MacColl.

Bogdanov, Yu. S. On the theory of systems of linear differential equations. Dokl. Akad. Nauk SSSR (N.S.) **104** (1955), 813–814. (Russian)

This paper deals with the characteristic numbers of Lyapunov (=c.n.). Take the system

(1) $$\dot{x}_i=\sum p_{ij}(t)x_j\ (i, j=1, 2, \cdots, n),$$

where the p_{ij} are continuous and bounded for $t\geqq t_0$. Let $\lambda_1, \cdots, \lambda_n$ in decreasing order of magnitude be the c.n. of some normal solution in the sense of Lyapunov and set

$$S=\sum\lambda_i,\ \sigma=-S-\liminf_{t\to+\infty}\frac{1}{t}\int_{t_0}^{t}\sum p_{ii}dt.$$

It is known that $\sigma \geq 0$, and if $\sigma=0$ the system is said to be regular. The following properties are stated without proof.

I. $$\sigma \leq \limsup_{t\to+\infty} \frac{n}{2t}\int_{t_0}^{t} \Sigma\,(|p_{ij}|+|p_{ji}|)dt - \liminf_{t\to+\infty} \frac{1}{t}\int_{t_0}^{t} \Sigma\, p_{ii}dt.$$

II. If $\lambda_1', \cdots, \lambda_n'$ are in increasing order the c.n. (for a normal solution) of the system adjoint to (1) then $0 \geq \lambda_i+\lambda_i' \geq -\sigma$. Hence if (1) is regular $\lambda_i+\lambda_i'=0$. III. Consider now

(2) $$\dot{y}=\Sigma\,(p_{ij}+q_{ij})y_j.$$

If the c.n. of the q_{ij} exceed σ then (2) has the same c.n. as (1). On the basis of III and using for (2) a piecewise linear system the author describes a construction for the c.n. λ_i.
S. Lefschetz (Mexico, D.F.).

Anke, Klaus. Eine neue Berechnungsmethode der quadratischen Regelfläche. Z. Angew. Math. Phys. 6 (1955), 327–331.

Let $p=d/dt$ denote the differential operator with respect to time t and consider the operator polynomial $f(p)=a_0p^n+a_1p^{n-1}+\cdots+a_n$ with real coefficients a_k. The differential equation $f(p)y(t)=0$ is assumed to admit stable solutions $y(t)$ only, and the author proposes to calculate the integral $J_0=\int_0^\infty y^2dt$ in terms of the coefficients a_k and the initial values $q_i=p^iy(0)$. To this end he multiplies the differential equation by $p^ky(t)$, integrates from zero to infinity and obtains a set of formulae for different values of k which permit calculation of J_0 by multiplying and inverting known matrices. The author admits that the problem was already considered and solved by various other authors [H. Sartorius, Dissertation, Stuttgart, 1945; P. Hazebroek and B. L. van der Waerden, Trans. A.S.M.E. 72 (1950), 309–315; MR 11, 666; H. Bückner, Quart. Appl. Math. 10 (1952), 205–213; MR 14, 145], but he claims that his method does not require the "explicit" knowledge of the solution $y(t)$. This is not so. Bückner's paper cited above gives an explicit formula for J_0 without using a representation of $y(t)$ in terms of exponential functions. The author also disregards that his only practical example referring to the minimization of J_0 under the conditions $a_0=a_n=1$, $q_0=1$, $q_1=q_2=\cdots=q_{n-1}=0$, was already discussed in Bückner's paper. *H. F. Bückner* (Schenectady, N.Y.).

César de Freitas, A. Sur les distributions qui interviennent dans le calcul symbolique des électrotechniciens (cas des circuits à constantes concentrées). Univ. Lisboa. Revista Fac. Ci. A. (2) 3 (1954–1954), 279–310.

Consider the equation

$$F(D)T=a_nD^nT+\cdots+a_1DT+a_0T=U$$

with constant coefficients and U a given distribution on $-\infty<x<\infty$. This paper gives a simplified discussion of just that part of distribution theory required to solve this equation when U is a Heaviside distribution; this means: in every finite interval, U can be expressed as a finite linear combination of a summable function and derivatives (arbitrary orders) of Dirac delta functions. U is said to be of class $\geq n$ if D^nU is also a Heaviside distribution. If U is of class ≥ 1, then $U(a^+)$, $U(a^-)$ are defined at each $x=a$.

Suppose T is Heaviside of class $\geq n$; then T is a solution if and only if

$$1^+F(D)T=1^+U \text{ and } 1^-F(D)T=1^-U,$$

i.e.,

(*) $$F(D)(1^+T)=1^+U+\sum_{i=0}^{n-1}A_i\delta^{(i)},$$

(**) $$F(D)(1^-T)=1^-U-\sum_{i=0}^{n-1}A_i\delta^{(i)},$$

where $1^+(x)=1$ for $x\geq 0$, $=0$ for $x<0$; $1^-(x)=0$ for $x\geq 0$, $=1$ for $x<0$ and the constants A_i depend on the (arbitrary) values of $T(0^-), \cdots, T^{(n-1)}(0^-)$. Now (*) and (**) have unique solutions (only distributions with support bounded on the left (or right) are considered) and these solutions can be expressed by the usual symbolic formulae. This gives the general solution $T=1^+T+1^-T$ (with n parameters $T(0^-), \cdots, T^{(n-1)}(0^-)$) since it is finally shown that every solution T is necessarily a Heaviside distribution of class $\geq n$. *I. Halperin* (Kingston, Ont.).

Foiaş, Ciprian; Gussi, George; and Poenaru, Valentin. Sur le problème polylocal pour les équations différentielles linéaires du second ordre. Acad. R. P. Romîne. Bul. Şti. Secţ. Şti. Mat. Fiz. 7 (1955), 699–721. (Romanian. Russian and French summaries)

Let y be a solution of a linear, homogeneous differential equation with real coefficients and let x_1, x_2 be two real roots of $y(x)$. De La Vallée Poussin has shown [J. Math. Pures Appl. (9) 8 (1929), 125–144] that there exists a constant h, depending only on the coefficients of the differential equation, such that $|x_1-x_2|<h$ implies $y(x)\equiv 0$. The author studies the particular case of the second-order differential equation $y''+A(x)y=0$, with $A(x)$ a continuous function. He indicates an algorithm for the computation of h with any desired accuracy. The method consists in proving first that h is a continuous functional with respect to $A(x)$. This permits one to approximate $A(x)$ uniformly over the closed interval (a, b), by its sequence of Bernstein polynomials and to approximate two independent solutions also by polynomials of two sequences, $P_k(x)$ and $Q_k(x)$. For a given k, the least distance between consecutive zeros of $y(x)=C_1P_k(x)+C_2Q_k(x)$ is the common root h of

$$\frac{P_k(x+h)}{Q_k(x+h)}=\frac{P_k(x)}{Q_k(x)}$$

and of

$$\frac{d}{dx}\left\{\frac{P_k(x+h)}{Q_k(x+h)}\right\}=\frac{d}{dx}\left\{\frac{P_k(x)}{Q_k(x)}\right\}.$$

Eliminating x, one obtains an equation in h; its smallest root r such that the root x of $y(x)$ and $x+r$ both belong to (a, b) is then taken as the approximation h_k of h. A very detailed investigation is made, in order to ascertain that the value h_k so obtained satisfies $|h-h_k|<\varepsilon$, with any preassigned $\varepsilon>0$. It is asserted (following Joukowsky) that if $A(x)$ is periodic, of period ω, the solutions of the differential equation stay bounded at infinity, provided that $\omega<h$. *E. Grosswald* (Philadelphia, Pa.).

Sandor, Ştefan. Sur les équations différentielles linéaires d'ordre supérieur aux coefficients presque périodiques. Acad. R. P. Romîne. Bul. Şti. Secţ. Şti. Mat. Fiz. 7 (1955), 329–346. (Romanian. Russian and French summaries)

Consider the equation $x^{(n)}+p_1(t)x^{(n-1)}+\cdots+p^n(t)x=0$ where

$$p_j(t)=\mu_j+\sum_{k=1}^{\infty} a_{kj}e^{i\omega_k t} \quad (j=1, \cdots, n).$$

$\alpha_{kj} \geq \beta > 0$ $(j=1, \cdots, n;\ k=1, 2, \cdots)$, and $\sum_{k=1}^{\infty} |a_{kj}| = M_j < \infty$ $(j=1, \cdots, n)$. Let $i\alpha$ be a q-fold root of equation (*) $k^n + \mu_1 k^{n-1} + \cdots + \mu_n = 0$ such that the numbers $i(\alpha + \sum_{k,j} l_{kj}\alpha_{kj})$ (l_{kj} non-negative integers) are not roots of (*) and do not cluster at zero. Then the differential equation has q solutions of the form

$$x(t) = e^{i\alpha t} \sum_{j=0}^{r} \Phi_{rj}(t) t^{r-j} \quad (r=0, 1, \cdots, q-1),$$

where

$$\Phi_{rj}(t) = \sum_{k=0}^{\infty} b_{kj} e^{i\beta_{kl}t}, \ \beta_{rl} = \sum_{k,j} l_{kj}\alpha_{kj}, \ \beta_{00} = 0.$$

This generalizes a result obtained by Putnam and Wintner [Amer. J. Math. **73** (1951), 792–806; MR **13**, 557], and partially generalizes a result given by A. Halanaĭ [Dokl. Akad. Nauk SSSR (N.S.) **88** (1953), 419–422; MR **15**, 311]. The proof is based on the method used by Halanaĭ. *E. A. Coddington* (Copenhagen).

Sandor, Ştefan. Les équations différentielles linéaires non homogènes à coefficients presque-périodiques et les équations quasi linéaires à petit paramètre. Acad. R. P. Romîne. Bul. Şti. Secţ. Şti. Mat. Fiz. **7** (1955), 683–698. (Romanian. Russian and French summaries)

The author considers the differential equation

$$L(x) \equiv \sum_{j=0}^{n-1} p_j(t) x^{(n-j)} + p_n(t)x = f(t) + \mu F(t, x, \mu),$$

where $p_0(t) \equiv 1$; $p_j(t) = \mu_j + \sum_{k=1}^{\infty} a_{kj} e^{i\alpha_{kj}t}$, $j=1, \cdots, n$; $\alpha_{kj} \geq \beta > 0$; $\sum_k |a_{kj}|$ finite. The distinct zeros of the polynomial $\alpha^n + \sum_{j=1}^{n} \mu_j \alpha^{n-j}$ are denoted $\alpha_1, \cdots, \alpha_m$, and it is assumed that the closure of all finite linear combinations $\sum_{k,j} l_{kj}\alpha_{kj}$ (l_{kj} integer) does not contain any of the numbers $i(\alpha_r - \alpha_s)$. The functions $f(t)$ and $F(t, x, \mu)$ are almost periodic in t. The function $F(t, x, \mu)$ may depend on x', $\cdots, x^{(n-1)}$ and it satisfies a Lipschitz condition of order 1 in these variables. A further condition, the author's condition B, which cannot be stated briefly, concerns some functions and coefficients which enter into the developments of the solutions. The author's criterion c states that condition B is satisfied if there exists a number δ such that $\gamma = \max(\operatorname{Im} \alpha_s + \delta) \leq \beta$ and the spectrum of $f(t)$ is situated to the right of γ. The principal results are: If B is satisfied and $\mu = 0$, the equation has at least one almost periodic solution. If no α_s is purely imaginary, the equation has one and only one almost periodic solution. If, further, every α_s has positive real part and $\mu = 0$, every other solution will tend asymptotically to the almost periodic solution. The almost periodicity of $f(t)$ and $F(t, x, \mu)$ is not mentioned in the summaries, and in the French summary the word "spectrum" is missing in the formulation of the criterion c.

H. Tornehave (Copenhagen).

Gorbunov, A. D. On some properties of solutions of systems of ordinary linear differential equations. Vestnik Moskov. Univ. **7** (1952), no. 12, 3–16. (Russian)

The author considers real nonhomogeneous linear differential systems (1) $dy/dt = L(t)y + f(t)$, where $y = (y_1, \cdots, y_n)$, $f = (f_1, \cdots, f_n)$ are $n \times 1$ matrices, $L = \|l_{ih}\|$ is an $n \times n$ matrix, the coefficients l_{ih} are supposed to be continuous in $0 \leq t < +\infty$, and the functions f_i continuously differentiable. By y', f' are denoted the same vectors y, f considered as $1 \times n$ matrices. In previous papers [same Vestnik **8** (1953), no. 9, 49–55; Moskov. Gos. Univ. Uč. Zap. **165**, Mat. **7** (1954), 39–78; MR **17**, 37; **16**, 475] the author studied the homogeneous system (2) $dx/dt = L(t)x$ and discussed the existence of a real symmetric quadratic form $G(t; x) = x'Ax = \sum a_{ih}(t) x_i x_h$ connected with Lyapunov stability of system (2). In the present paper asymptotic evaluations are obtained for the solutions of system (1). If $g(t; y) = y'(dA/dt + L'A + AL)y$, $h(t; y) = f'Ay + y'Af$, let $N(t) = \max g(t; y)$, $Q(t) = \max h(t; y)$ for all y with $G(t; y) = 1$. If y is the solution of (1) with $y = y^0$ at $t = t_0$, then the following inequalities hold: $|y_s| \leq [G(t_0; y^0) D_s(t) \exp \int_{t_0}^{t} N(\tau)d\tau]^{\frac{1}{2}} +$
$\frac{1}{2} D_s^{\frac{1}{2}}(t) \int_{t_0}^{t} Q(\xi)d\xi \exp \frac{1}{2}\int_{t_0}^{t} N(\tau)d\tau$ $(s=1, \cdots, n)$,

where $D_s(t) = A_{n-1}^{(s)}(t) A_n^{-1}(t)$ is the usual quotient of minors of A as in the second paper cited above.

L. Cesari (Lafayette, Ind.).

Breus, K. A. On asymptotic solution of linear differential equations with periodic coefficients. Dopovidi Akad. Nauk Ukrain. RSR **1955**, 415–418. (Ukrainian. Russian summary)

Let $d^2x/dt^2 + [A + \psi(\omega t)]x = 0$ be an equation in vector-matrix form, where x is an n-dimensional vector, A is a constant symmetric matrix, ω is a large parameter, and $\psi(\omega t)$ a periodic matrix. The author gives formal solutions of this equation as a power series in $\varepsilon = \omega^{-1}$. *S. Kulik.*

Kostomarov, D. P. On asymptotic behavior of solutions of linear differential equations of second order in the neighborhood of an irregular singular point. Dokl. Akad. Nauk SSSR (N.S.) **103** (1955), 759–762 (Russian)

The differential equation (1) $d^2w/dz^2 - Q(z)w = 0$, z complex, is transformed into the analogous equation $d^2w_1/dz_1^2 - Q_1(z_1)w_1 = 0$ by $z_1 = \int_{z_0}^{z} Q^{\frac{1}{2}}(\xi)d\xi$, $w_1 = Q^{\frac{1}{4}}w$, where $Q_1 = 1 - (5Q'^2 - 4QQ'')(16Q^3)^{-1}$. By indefinite repetition of this transformation we obtain the equations $d^2w_{n+1}/dz^2_{n+1} - Q_{n+1}(z_{n+1})w_{n+1} = 0$, where $w_{n+1} = F_n^{\frac{1}{4}}w$, $z_{n+1} = \int_{z_0}^{z} F_n(\xi)d\xi$, $F_n = QQ_1 \cdots Q_n$. If $F_n \underset{n\to}{\rightrightarrows} F$ uniformly in a region R, then equation (1) has solutions of the form $w_i(z) = A_i F^{-\frac{1}{4}} \cdot \exp(-1)^i \int_{z_0}^{z} F^{\frac{1}{2}} d\xi$ $(i=1, 2)$, where A_1, A_2 are constant. Besides the proof of this statement the paper contains a brief discussion of these solutions at $z = \infty$. *L. Cesari.*

Schwarz, Hans Rudolf. Critère de stabilité pour des systèmes à coefficients constants. C. R. Acad. Sci. Paris **241** (1955), 15–16.

The author gives a reduction of a real square matrix to a form from which the number of characteristic values with negative real parts and the number with positive real parts can be determined without calculating the characteristic equation. Use is made of a criterion of Wall [Analytic theory of continued fractions, Van Nostrand, New York, 1948, p. 182 ff.; MR **10**, 32]. *W. S. Loud.*

Schwarz, Hans Rudolf. Critère de stabilité pour des systèmes d'équations différentielles à coefficents constants complexes. C. R. Acad. Sci. Paris **242** (1956), 325–327.

The reduction in the paper reviewed above is extended to matrices with complex entries. *W. S. Loud.*

Aminov, M. Š. On a method for obtaining sufficient conditions for stability of unsteady motion. Prikl. Mat. Meh. **19** (1955), 621–622. (Russian)

The author presents some sufficient conditions for stability in the case where some characteristic roots have

zero real parts. His method utilizes the Lyapunov functions. *R. Bellman* (Santa Monica, Calif.).

Lidskiĭ, V. B.; and Neĭgauz, M. G. On criteria for stability of a system of differential equations with periodic coefficients. Prikl. Mat. Meh. **19** (1955), 625–627. (Russian)

The author considers the problem of determining sufficient conditions upon n, λ, and $\phi(t)$ to ensure the boundedness of all solutions of the vector-matrix equation $y''+(n^2+\lambda\phi(t))y=0$, where n and λ are scalars and $\phi(t)$ is a symmetric, periodic matrix function. The results are related to those of M. G. Krein [Dokl. Akad. Nauk SSSR (N.S.) **73** (1950), 445–448; Prikl. Mat. Meh. **15** (1951), 323–348; MR **12**, 100; **13**, 348]. *R. Bellman.*

Yakubovič, V. A. On systems of differential equations of canonical form with periodic coefficients. Dokl. Akad. Nauk SSSR (N.S.) **103** (1955), 981–984. (Russian)

Consider $\dot{x}=JH(t)x$, J, H real matrices of order $2k$, $J=\begin{pmatrix} 0 & E \\ -E & 0\end{pmatrix}$, H symmetric and periodic. The case $k>1$ is studied here and is shown to be similar to the case $k=1$ previously considered by the author [Mat. Sb. N.S. **37(79)** (1955), 21–68; MR **17**, 483]. The space L of symmetric periodic matrices with norm $\int_0^\omega \|H(t)\|dt$ splits into a denumerable set of domains of stability $O_n^{(\mu)}$ ($n=0, \pm 1, \pm 2, \cdots$) the index μ taking 2^k values, a set of instability N (for $k=1$ this set is not connected) and a common boundary Γ_0. Theorems of comparison of the following type are stated: If $H_1, H_2 \in O_n^{(\mu)}$, $H_1 \leq H \leq H_2$, then $H \in O_n^{(\mu)}$; a similar theorem is false for the set N but holds true again if restricted to certain parts $N_n^{(\mu)}$ of N. From these main theorems a number of criteria for stability and instability may be derived. For instance, if $H=H_0+H_1$, $H_1=\operatorname{diag}[h_1(t), \cdots, h_k(t), h_1(t), \cdots, h_k(t)]$, if $h_1^0(t)$ and $h_2^0(t)$ are the smallest and largest eigenvalues of H_0, then H is stable if integers m_{ij} exist such that

$$2m_{ij}\pi<\int_0^\omega (h_i+h_j+2h_1^0)dt \leq \int_0^\omega (h_i+h_j+2h_2^0)dt < 2(m_{ij}+1)\pi \quad (i, j=1, \cdots, k).$$

Many other results are stated which it is difficult to summarize. No proofs are given. *J. L. Massera.*

Putnam, C. R. Note on some oscillation criteria. Proc. Amer. Math. Soc. **6** (1955), 950–952.

The author deduces criteria in terms of the functions $F(t)=\int_0^t f(s)ds$ and $G(t)=t^{-1}\int_0^t F(s)ds$ for (1) $x''+f(t)x=0$ to be oscillatory (for large t). For example, if $E(M, T)$ denotes the set of points t such that $t>T$ and $G(t)>M$ and if $\exp(M_nT_n)$ meas $E(M_n, T_n)\to\infty$ for some sequences M_n, T_n satisfying $M_n, T_n\to\infty$ as $n\to\infty$, then (1) is oscillatory. *P. Hartman* (Baltimore, Md.).

Friedrichs, K. O. Asymptotic phenomena in mathematical physics. Bull. Amer. Math. Soc. **61** (1955), 485–504.

The occurrence of discontinuities, shock, boundary layer, edge effect, skin effect, Stokes' phenomena, etc. are discussed from the physical and mathematical points of view in this excellent exposition based on the Gibbs lecture delivered by the author in 1954. Unsolved and partially solved problems are mentioned and a list of selected references is given. *N. Levinson.*

Szmydt, Z. Sur l'allure asymptotique des intégrales de certains systèmes d'équations différentielles non linéaires Ann. Polon. Math. **1** (1955), 253–276.

The present paper concerns the asymptotic behavior of the solutions of the non-linear system

(1) $dy/dt=Ay+f(t, y)$, $y=(y_1, \cdots, y_n)$, $f=(f_1, \cdots, f_n)$, $A=\|a_{ij}\|$, $t\geq 0$,

where a_{ij} are constants and the functions f_i of t and y are continuous. $\|y\|$ denotes the Euclidean norm. Consider the linear system (2) $dx/dt=Ax$ and denote by $+1$ the maximum of the exponents of the elementary divisors of the characteristic matrix $\|A-I\|$. The following main theorem is proved. I. If $\|f\|\leq h(t)\|y\|$, where $h(t)$ is a real-valued continuous function of t with $\int_0^{+\infty} h(t)dt<+\infty$, and if a uniqueness theorem holds, then for every solution $y^*(t)$ of (2) not identically zero there is at least one solution $y(t)$ of (1) with $\|y(t)-y^*(t)\|/\|y^*(t)\|\to 0$ as $t\to+\infty$. The integrals y, y^* then are said to be asymptotically associated. Various more stringent formulations are given of Theorem I and extensions to the comparison of the solutions of systems, both non-linear. Consistent use is made in the proofs of Ważewski's topological method. [For other papers on the same method see T. Ważewski, Ann. Soc. Polon. Math. **20** (1947), 279–313; Bull. Acad. Polon. Sci. Cl. III. **1** (1953), 3–5; MR **10**, 122; **15**, 222; F. Albrecht, ibid. **2** (1954), 315–318; MR **16**, 248; A. Pliś, ibid. **2** (1954), 415–418; MR **16**, 700; K. Tatarkiewicz, Ann. Univ. Mariae Curie-Skłodowska. Sect. A. **7** (1953), 19–81; MR **16**, 821.] Theorem I extends a previous result of S. Faedo [Ann. Mat. Pura Appl. (4) **26** (1947), 207–215; MR **10** 120] and of E. Levi [Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) **8** (1950), 465–470; **9** (1950), 26–31; MR **12**, 611, 827] who had proved an analogous result under the hypothesis $\|f(t, y_1)-f(t, y_2)\|\leq h(t)\|y_1-y_2\|$ by a particular analytical method. Theorem I extends also a previous result of N. Levinson [Duke Math. J. **15** (1948), 111–126; MR **9**, 509] for linear systems. *L. Cesari.*

Mikołajska, Z. Sur l'allure asymptotique des intégrales des systèmes d'équations différentielles au voisinage d'un point asymptotiquement singulier. Ann. Polon. Math. **1** (1955), 277–305.

This is another application of Ważewski's topological theory [see the above review for references]. Differential systems are considered

(1) $dZ/dt=f(t, z)$, $Z=(z_1, \cdots, z_n)$ $(a<t<b)$,

where f is a real continuous vector function of t and Z in the open set Ω which is the product of $a<t<b$, and of an open set θ of the n-dimensional real $(z_1, \cdots, z_n)$-space. If $z=\phi(t)$, $\alpha<t<\beta$, is an integral of (1) then we denote by $\phi([t_0, \varepsilon])$ the subset of all points $z=\phi(t)\in\theta$ with $t_0\leq t<\beta$. An integral $\phi(t)$ is said to be asymptotic relative to the open subset $\omega\subset\theta$, or to the point $z_0\in\theta$ provided $\beta=b$ and $\phi([t_0, b])\subset\omega$, or $\beta=b$ and $\phi(t)\to z_0$ as $t\to b-0$ (briefly, asymptotic rel. to ω, or rel. to z_0). A point z_0 is said to be asymptotically strongly singular (a.s.s.) provided there is a neighborhood ω of z_0 in θ such that the family of all integrals asymptotic rel. to ω is not empty and coincides with the family of all integrals asymptotic rel. to z_0. Both the following two problems are discussed: (a) sufficient conditions in order that a point z_0 be a.s.s.; (b) evaluations of the dimension of the family of the integrals which are asymptotic rel. to a point z_0 supposed to be a.s.s.. This is a theorem concerning problem (a), where z_0 is taken to be $z_0=0$ and is supposed a.s.s.. Suppose system (1) is written in the form

(2) $dX/dt=F(t, X, Y)$, $dY/dt=G(t, X, Y)$,

$X=(x_1, \cdots, x_p)$, $Y=(y_1, \cdots, y_q)$, $p+q=n$, $p\geq 0$, $q\geq 0$,

and suppose a uniqueness theorem holds. Given $\varepsilon>0$, $\delta>0$, denote by S and E the sets $S=[|X|\leq\gamma,\ |Y|\leq\delta,\ t_0<t<b]$, $E[|X|\leq\gamma,\ |Y|=\delta,\ t_0<t<b]$ and by XF, YG the real scalar products of X and F, of Y and G. Suppose that for given γ, $\delta>0$, we have $XF>0$ in S and $YG<0$ in E (where if $p=0$ [$q=0$] only the second [first] relation is considered). Then the family of the solutions asymptotic rel. to $z_0=0$ is at least q-dimensional [has at least one element if $q=0$]. *L. Cesari* (Lafayette, Ind.).

Hahn, Wolfgang. Über Stabilität bei nichtlinearen Systemen. Z. Angew. Math. Mech. **35** (1955), 459–462. (English, French and Russian summaries)

Let $\dot{x}=Px$ and let $V(x)=x'Ax$ be such that $\dot{V}(x)=-x'Bx$ where A, B are positive definite. Then $y=0$ is uniformly asymptotically stable in the large relative to $\dot{y}=(P+Q)y$ if and only if $\dot{V}(y)=y'Cy$ is negative definite [see, e.g., A. Liapounoff, Problème général de la stabilité du mouvement, Princeton, 1947, pp. 255–262, 273–277; MR **9**, 34]. Known criteria for the negative definiteness of C yield inequalities for the elements of Q. The author proposes a method for obtaining these inequalities explicitly and shows that they are of degree $\leq 2r$ if r rows of Q are not identically zero. Extensions to systems $\dot{z}=Pz+Z(z)$ where $Q_1z<Z<Q_2z$ are evident. *H. A. Antosiewicz* (Washington, D.C.).

Lyaščenko, N. Ya. On asymptotic stability of solutions of nonlinear systems of differential equations. Dokl. Akad. Nauk SSSR (N.S.) **104** (1955), 177–179. (Russian)

The author considers systems of the form (1) $dy/dt=A(t)y+\varphi(t)+f(t, y)$, where y, $\varphi(t)$, $f(t, y)$ are $n\times 1$ matrices, and $A(t)$ a $n\times n$ matrix. Let $K>0$ be a constant related to the matrix $A(t)$ by a set of conditions which are given below and which had been discussed by the author in a previous paper [same Dokl. (N.S.) **96** (1954), 237–239; MR **16**, 133] concerning the homogeneous system (2) $dx/dt=A(t)x$. Suppose $\varphi(t)$ bounded and continuous in $[0, +\infty)$, $f(t, y)$ continuous in t and y, $f(t, 0)=0$, $|f(t, y)-f(t, y')|\leq L|y-y'|$, where L is a constant. Denote by $\varrho_i(t)$ $(i=1, \cdots, n)$ the roots (functions of t) of the characteristic equation $\det\|\varrho E-A(t)\|=0$. The author proves that if $\operatorname{Re}\varrho_i(t)\leq-\gamma$ $(\gamma>0)$ for all $0\leq t<+\infty$ $(i=1, \cdots, n)$, and $L<4^{-1}K^{-1}\gamma(1-\sigma)$ for some $0<\sigma<1$; then system (1) has a solution $y(t)$ which is asymptotically stable in the sense of Lyapunov. The set of conditions mentioned above and relative to (2) can be stated briefly as follows: $|A(t)|\leq M$, $|A(t)|-A(t')|\leq\delta|t-t'|$, $\delta\leq(\exp\gamma T)/2a^2T^2$, where $a=2^n\lambda/\gamma^n$, $\lambda=2^{n-1}M^n(n-1)^{(n-1)/2}$, $T=4\gamma^{-1}\ln|2a|$, $|x(t)|\leq K|x(0)|\exp(-\gamma t)$ $(t\geq 0)$. [For the consistent use of characteristic roots $\varrho_i(t)$, functions of t, besides N. Ya. Lyaščenko (loc. cit.) see B. P. Demidovič, ibid. **72** (1950), 1005–1008; MR **12**, 181; R. Bellman, Duke Math. J. **14** (1947), 83–97; MR **9**, 35; N. Levinson, Amer. J. Math. **68** (1946), 1–6, MR **7**, 381; L. Cesari, Ann. Scuola Norm. Sup. Pisa (2) **9** (1940), 163–186; MR **3**, 41.] *L. Cesari.*

Szmydt, Z. Sur les systèmes d'équations différentielles dont toutes les solutions sont bornées. Ann. Polon. Math. **2** (1955), 234–236 (1956).

A remark concerning the differential system (1) $x'=A(t)x+f(t, x)$, where $A(t)$ is a $n\times n$ matrix with continuous coefficients and $f(t, x)$ is a continuous vector function of t and x, $0\leq t<\infty$. Let $Y(t)$, $t\geq 0$, denote a fundamental solution of the linear system $y'=A(t)y$. If (a) $\|Y(t)\|\leq M$, $0\leq t<\infty$; (b) $\|f(t, x)\|\leq g(t)\|x\|$, $\int_0^{+\infty}g(t)dt<+\infty$; (c) $\|Y(t)Y^{-1}(\tau)\|\leq N$ for all $0\leq\tau\leq t$, then all solutions of (1) are bounded. II. If (a), (b) hold, and (d) $f(t, Y(t)z)=Y(t)f(t, z)$ for all $t\geq 0$ and vectors z, then all solutions of (1) are bounded. I is an immediate consequence of the reduction of (1) to an integral equation. II follows by a remark of H. Weyl [Amer. J. Math. **68** (1946), 7–12; MR **7**, 382]. The norm above, $\|\ \|$, is the usual square root of the integral-squares of the elements. *L. Cesari.*

Szmydt, Z. On the degree of regularity of surfaces formed by the asymptotic integrals of differential equations. Ann. Polon. Math. **2** (1955), 294–313 (1956).

The author studies the autonomous differential system (1) $dy/dt=f(y)$, $y=(y_1, \cdots, y_n)$, $f=(f_1, \cdots, f_n)$, for which the origin is the singular point, i.e. $f(0)=0$. Also, it is assumed that the characteristic roots of the matrix $C=\|c_{ij}\|$, c_{ij} being the partial derivative of f_i with respect to y^j at the point $y=0$, have the real parts different from zero and two of them have real roots of opposite signs. Let S denote the subset of E_n made up of all trajectories $y=y(t)$ of system (1) tending to the singular point as $t\to+\infty$. I. G. Petrovskiĭ proved [Mat. Sb. **41** (1934), 107–156] that S is a continuous manifold under general conditions of f. M. Martin [Bull. Amer. Math. Soc. **46** (1940), 475–481; MR **2**, 50] proved that S is an analytic manifold if f is analytic. In the present paper the author proves that S is a manifold of class C^p if f is of class C^p, $p\geq 1$. *L. Cesari* (Lafayette, Ind.).

Blehman, I. I. On the problem of stability of periodic solutions of quasilinear systems with many degrees of freedom. Dokl. Akad. Nauk SSSR (N.S.) **104** (1955), 809–812. (Russian)

The problem discussed is the stability for small μ of the periodic solutions of

$$(1)\qquad \dot{x}=Ax+f(t)+\mu F(x, \mu, t),$$

where x, f, F are n-vectors and A is a constant matrix. Moreover, F is analytic and f, F have uniformly convergent Fourier series (period 2π) in a certain domain G where F is analytic. Furthermore, it is assumed that for $\mu=0$ there is a periodic solution. The proof of existence, uniqueness and asymptotic stability of the periodic solutions for μ small, follow a standard pattern. [Reviewer's observation: The author does not seem to be aware of the closely related paper by Coddington and Levinson, Contributions to the theory of nonlinear oscillations, v. 2, Princeton, 1952, pp. 19–35; MR **14**, 981.] *S. Lefschetz* (Mexico, D.F.).

Kononenko, V. O. On nonlinear oscillations in systems with varying parameters. Dokl. Akad. Nauk SSSR (N.S.) **105** (1955), 229–232. (Russian)

The systems dealt with are reducible to the form

$$\dot{x}_h=\sum q_{hk}(\theta, \tau)x_k+\varepsilon V_h(x_1, x_2, \theta, \tau, \varepsilon)\quad (h, k=1, 2),$$

where ε is small, $\tau=\varepsilon t$ is a slowly varying time, $\dot{\theta}=\nu(\tau)$ is approximately constant and $q_{hk}(\theta+2\pi, \tau)=q_{hk}(\theta, \tau)$. One first assumes τ constant and solves the linear system corresponding to $V_1=V_2=0$. In the process the characteristic exponents are assumed pure complex. To obtain the periodic solutions of the complete system one applies a method of successive approximations (in powers of ε) with elimination of the secular terms at each step (Gylden-Lindstedt process of astronomy). The author observes that the method (strictly asymptotic) is applicable even when $\tau=0$, i.e. to non-linear periodic systems. *S. Lefschetz* (Mexico, D.F.).

Leimanis, Eugene. On a theorem of Poincaré and Malkin. Trans. Roy. Soc. Canada. Sect. III. (3) **49** (1955), 39–48.

From the author's introduction: "The theorem in question is concerned with the existence of periodic solutions of a non-autonomous system of order n, containing a small parameter μ and being such that the generating system admits an infinity of periodic solutions depending upon k ($<n$) arbitrary parameters. The case of a single parameter was considered by Poincaré [Les méthodes nouvelles de la mécanique céleste, t. 1, Gauthier-Villars, Paris, 1892, p. 84] and the general case with $1<k$ ($<n$) parameters by Malkin [The methods of Lyapunov and Poincaré in the theory of nonlinear oscillations, Gostehizdat, Moscow-Leningrad, 1949, p. 23; MR **12**, 28]. In this paper we shall consider an exceptional case of Malkin's theorem, when certain equations are satisfied identically." *M. Zlámal* (Brno).

Halanay, A. Points singuliers et solutions périodiques. Acad. R. P. Romîne. Bul. Şti. Secţ. Şti. Mat. Fiz. **7** (1955), 319–324. (Romanian. Russian and French summaries)

The author considers the real system

$$(1)\quad \begin{aligned} dx/dt&=Q(x, y)+\mu X(x, y, t),\\ dy/dt&=P(x, y)+\mu Y(x, y, t), \end{aligned}$$

where P, Q are homogeneous polynomials of the same degree and X, Y are analytic functions of x, y, t and periodic in t. Denote by $f(k)$ the function

$$f(k)=P(1, k)/Q(1, k)$$

and suppose that: (1) the equation $f(k)=k$ has at least two real roots; (2) $f'(k)<0$. Then for all μ sufficiently small in absolute value, system (1) has periodic solutions. Ważewski's topological theory is used [Ann. Soc. Polon. Math. **20** (1947), 279–313; MR **10**, 122] and a theorem of Massera [Duke Math. J. **17** (1950), 457–475; MR **12**, 705]. *L. Cesari* (Lafayette, Ind.).

Barbălat, I. Remarques sur la note "Points singuliers et solutions périodiques". Acad. R. P. Romîne. Bul. Şti. Secţ. Şti. Mat. Fiz. **7** (1955), 325–328. (Romanian. Russian and French summaries)

The present paper extends the result of the paper reviewed above concerning systems of the form

$$(1)\quad \begin{aligned} dx/dt&=Q(x, y)+\mu X(x, y, t),\\ dy/dt=dy/dt&=P(x, y)+\mu Y(x, y, t), \end{aligned}$$

where now Q, P are supposed to be real functions of x and y, holomorphic in a neighborhood of the origin $x=y=0$. In the present paper results and notations of S. Lefschetz [Contributions to the theory of nonlinear oscillations, v. II, Princeton, 1952, pp. 61–73; MR **14**, 557] are used. If the analytic function defined by $R(x, y)\equiv xQ=yP=0$ has only single branches through the origin which have no contacts there and divide a neighborhood of the origin in at least four sectors all of the type I in the sense of S. Lefschetz (loc. cit.), then system (1) has periodic solutions for μ sufficiently small in absolute value. *L. Cesari* (Lafayette, Ind.).

Albrecht, Felix. Points singuliers et solutions périodiques. Com. Acad. R. P. Romîne **5** (1955), 1035–1040. (Romanian. Russian and French summaries)

This paper concerns a more straightforward application of the author's topological method to the question of existence of periodic solutions of the system

$$(1)\quad \begin{aligned} dr/dt&=Q(x, y)+\mu X(x, y, t),\\ dy/dt&=P(x, y)+\mu Y(x, y, t), \end{aligned}$$

already discussed in the two papers reviewed above. Here X, Y are functions of class C' of x, y, t, periodic in t with period ω. Also, Q and P are supposed to be real analytic functions of x and y in a neighborhood Ω of the origin $x=y=0$ which is an isolated singularity. If all real branches through the origin defined by $R(x, y)=xQ+yP=0$ are single, and the set Ω has at least four points of "external sliding" in the sense of Ważewski, and the points p of internal sliding satisfy the condition

$$(2)\quad T^{\epsilon}(p)\cap \operatorname{Ext}(\Omega)\neq\emptyset,$$

then system (1) has at least one periodic solution for every μ in absolute value sufficiently small. In (2) $\emptyset$ is the null set, $\epsilon=\pm1$, and $T^{\pm}(p_0)$ denotes the projection on the xy-plane of the half-trajectory $x=x(t)$, $y=y(t)$, $t\geqq t_0$ [$t\leqq t_0$] through p_0 at $t=t_0$. Finally, in (2) we have $\epsilon=+1$, or $\epsilon=-1$, according as $dr/dt>0$, or <0, $r=(x^2+y^2)^{\frac{1}{2}}$, this sign being constant in each sector around the origin defined in Ω by the real branches above. *L. Cesari.*

Aymerich, Giuseppe. Cicli di prima e di seconda specie di un sistema meccanico autosostenuto impulsivamente. Rend. Sem. Fac. Sci. Univ. Cagliari **25** (1955), 26–36.

The author considers the system of differential equations $dx/dt=y$, $dy/dt=-2\lambda y-\sin x$, together with the condition

$$\lim_{x\to 2n\pi+} y-\lim_{x\to 2n\pi-} y=\alpha\quad (n=0, \pm1, \pm2, \cdots),$$

α and λ being constants. The equations are interpreted as the equations of motion of a pendulum, capable of performing complete revolutions, which is subjected to an impulse each time it passes through its configuration of stable equilibrium. The methods of phase-plane analysis are used to obtain the following results. (1) If $4\pi\lambda\geqq\alpha+(\alpha^2+4\pi)^{\frac{1}{2}}$, there exists a unique stable oscillatory periodic motion. (2) If $4\pi\lambda\geqq\alpha\geqq\pi\lambda+\pi(\lambda^2+1)^{\frac{1}{2}}$, there exists a periodic rotatory motion. *L. A. MacColl.*

★ **Uno, Toshio. On subharmonic resonance.** Proceedings of the Second Japan National Congress for Applied Mechanics, 1952, pp. 307–308. Science Council of Japan, Tokyo, 1953.

The author considers the system of differential equations

$$\frac{dx}{dt}=X(x, y)+kp(t),\quad \frac{dy}{dt}=Y(x, y)+kq(t),$$

assuming that p and q are periodic with period τ, and that when $k=0$ the system possesses a solution which is periodic with period τ_0. Topological methods are used to reach certain conclusions concerning the existence and stability of periodic solutions in the cases in which $|k|$ is small but not zero. The paper contains neither proofs nor precisely stated theorems; and the relations between this discussion and other work on similar problems are not elucidated. *L. A. MacColl* (New York, N.Y.).

★ **Shimizu, Tatsujiro. Subharmonics for non-linear differential equations.** Proceedings of the Third Japan National Congress for Applied Mechanics, 1953, pp. 421–423. Science Council of Japan, Tokyo, 1954.

In (1) $x''+cx+dx^3=e\sin\omega t$, let c, d, e be constants.

The author shows that if $d\neq 0$, then (1) has solutions with periods exceeding a given constant if $|c|\neq 0$ and $|e|$ are sufficiently small. For the case $c>0$, $d>0$, cf. Shimizu, Math. Japon. **2** (1951), 86–96, pp. 93–95 [MR **13**, 745].
P. Hartman (Baltimore, Md.).

★ **Shimizu, Tatsujiro. Resonant oscillations for some non-linear differential equations.** Proceedings of the Fourth Japan National Congress for Applied Mechanics, 1954, pp. 443–446. Science Council of Japan, Tokyo, 1955.

The author considers the differential equation

$$\ddot{x}+\varepsilon r\dot{x}+\alpha x+\varepsilon\beta x^3=F\cos\omega t,$$

assuming that ε, r, α, β, F, ω are positive constants, and that ε is small. He discusses relations between the value of F and the qualitative properties of the solutions. The methods employed are not rigorous, and no essentially new results are obtained. *L. A. MacColl.*

Urabe, Minoru. The least upper bound of a damping coefficient ensuring the existence of a periodic motion of a pendulum under constant torque. J. Sci. Hiroshima Univ. Ser. A. **18** (1955), 379–389.

The equation

$$\ddot{\theta}+\alpha\dot{\theta}+\sin\theta=\beta\quad(\alpha,\beta\geqq 0)$$

is studied by the general methods of the author [same J. **18** (1954), 183–219; MR **17**, 264]. The main purpose is to find the sup $\alpha(\beta)$ of the values α for which periodic solutions of the second kind exist. It is shown that $\alpha(\beta)=\infty$ for $\beta>1$ and $\alpha(\beta)$ is continuous to the left at $\beta=1$. A method for the computation of $\alpha(\beta)$ is devised which is carried through with an accuracy of 0.001.
J. L. Massera (Montevideo).

Honnorat, P. Sur une approximation dans le problème du bipendule. Publ. Sci. Univ. Alger. Sér. A. **1** (1954), 303–306 (1955).

The author criticizes a method of Pöschl [Atti 4° Congresso Un. Mat. Ital., Taormina, 1951, v. 2, Edizioni Cremonese, Roma, 1953, pp. 542–549; MR **15**, 69] for the approximate solution of problems concerning oscillations of nonlinear systems. He concludes that the method is inferior to well known classical methods.
L. A. MacColl (New York, N.Y.).

Mitrinovitch, Dragoslav S. Sur l'équation différentielle d'un problème d'hydrodynamique. C. R. Acad. Sci. Paris **241** (1955), 1708–1710.

It is shown how to solve the ordinary differential equation in two unknown functions, $w(z)$ and $v(z)$,

$$\frac{1}{w}\frac{d^2w}{dz^2}=h_3\frac{1}{v}\frac{d^2v}{dz^2}+h_2\frac{1}{v^2}+h_1,$$

where h_3, h_2, h_1 are constants, by assuming that $w=T(v)$, T being an undetermined function. The form of the solution for the case $h_3=1$, $h_2=0$ is also written down.
G. C. McVittie (Urbana, Ill.).

Haber, S.; and Levinson, N. A boundary value problem for a singularly perturbed differential equation. Proc. Amer. Math. Soc. **6** (1955), 866–872.

The authors study the solution, $y(x,\varepsilon)$, of the boundary-value problem

$$(1)\qquad \varepsilon y''=f(x,y,y',\varepsilon),\quad y(0)=a,\quad y(1)=b,$$

where the degenerate equation $f(x,u,u',0)$ has solutions

$$U(x)=\begin{cases} g(x), & 0\leqq x\leqq x_0,\quad g(0)=a,\\ h(x), & x_0\leqq x\leqq 1,\quad h(1)=b,\end{cases}$$

and $g(x_0)=h(x_0)$, but $g'(x_0)\neq h'(x_0)$. This situation arises if

$$f_{y'}(x,g,g',0)>0,\quad 0\leqq x\leqq x_0,$$
$$f_{y'}(x,h,h',0)<0,\quad x_0\leqq x\leqq 1.$$

In this case, there exists a unique solution to (1) for ε sufficiently small, which tends uniformly to $U(x)$ over $0\leqq x\leqq 1$, and whose derivative tends uniformly to $U'(x)$ over $0\leqq x\leqq x_0-\delta$, $x_0+\delta\leqq x\leqq 1$, $\delta>0$. The only requirements on $f(x,y,y',\varepsilon)$ are continuity of f, f_y, $f_{y'}$ for $0\leqq\varepsilon\leqq\varepsilon_0$, for all values of y, y' sufficiently close to U, U', and for $0\leqq x<x_0<x\leqq 1$. *G. E. Latta* (Stanford, Calif.).

Sears, D. B. Integral transforms and eigenfunction theory. II. Quart. J. Math. Oxford Ser. (2) **6** (1955), 213–217.

According to Weyl [cf. Kodaira, Amer. J. Math. **71** (1949), 921–945; MR **11**, 438] there is associated, with a self-adjoint operator H belonging to $x''+(\lambda+q(t))x=0$, a non-decreasing function $k(\lambda)$ and a unitary transformation $f(t)\sim F(\lambda)$ between functions of class L_2: $\int|f(t)|^2dt<\infty$ and of class l_2: $\int|F(\lambda)|^2dk(\lambda)<\infty$. In order to simplify a step in his proof [Quart. J. Math. Oxford Ser. (2) **5** (1954), 47–58; MR **15**, 959] of the Titchmarsh form of the Weyl expansion theorem, the author deduces, without approximating by Sturm-Liouville operators, the fact that if $\lambda F(\lambda)\in l_2$, then f is in the domain of H and $Hf\sim\lambda F(\lambda)$.
P. Hartman (Baltimore, Md.).

Glazman, I. M.; and Naĭman, P. B. On the convex hull of orthogonal spectral functions. Dokl. Akad. Nauk SSSR (N.S.) **102** (1955), 445–448. (Russian)

The problem $-y''+q(x)y=\lambda y$, $y'(0)=hy(0)$ on $0\leqq x<\infty$ is considered. Let φ be the solution of the differential equation satisfying $\varphi(0,\lambda)=1$, $\varphi'(0,\lambda)=h$. Then there exists a set V of non-decreasing spectral functions σ on $-\infty<\lambda<\infty$ ($\sigma(0)=0$, $\sigma(\lambda-0)=\sigma(\lambda)$) such that σ generates an isometric mapping $f\to F$ of $L^2(0,\infty)$ onto a subspace M_σ of $L^2(\sigma)$ via the formulas $F(\lambda)=\int_0^\infty f(x)\varphi(x,\lambda)dx$, $f(x)=\int_{-\infty}^\infty F(\lambda)\varphi(x,\lambda)d\sigma(\lambda)$. If $M_\sigma=L^2(\sigma)$ the function σ is said to be orthogonal. The set V is convex and closed, in the sense that if σ is a non-decreasing function such that $\sigma_n\to\sigma$ at the continuity points of σ, then $\sigma\in V$. Let N be the set of all functions τ of z which are regular in the upper half of the complex plane and have there a non-negative imaginary part. M. G. Krein [same Dokl. (N.S.) **89** (1953), 5–8; MR **15**, 316] has shown that there is a 1-1 correspondence between V and N defined by the formula

$$\frac{P_0(z)\tau(z)+P_1(z)}{Q_0(z)\tau(z)+Q_1(z)}=\int_{-\infty}^{\infty}\frac{d\sigma(\lambda)}{\lambda-z},$$

where P_0, P_1, Q_0, Q_1 are certain entire functions uniquely determined by the differential equation and boundary condition at zero. The authors state two lemmas which are then used to prove that the closed convex hull U of the orthogonal spectral functions is generated by those $\tau\in N$ of the form $\tau(z)=\tau^*(u(z))$, where τ^* is an arbitrary function in N and $u(z)=Q_0(z)/Q_1(z)$. The orthogonal spectral functions are extreme points of V, and $U\neq V$. The authors state conditions which allow identification of other extreme points, and this leads to the fact that V is the closure of the set of its extreme points.
E. A. Coddington (Copenhagen).

Hartman, Philip; and Wintner, Aurel. An inequality for the first eigenvalue of an ordinary boundary value problem. Quart. Appl. Math. 13 (1955), 324–326.

The boundary-value problem (P): $y''+g(x)y'+f(x)y=0$, $y(a)=0$, $y(b)=0$, is considered. Here f and g are real-valued, continuous on $a\leqq x\leqq b$, and, unless g' is not involved, it is assumed g' is continuous. It is proved that, if there exists a real constant c satisfying

(*) $\quad f(x)-cg'(x)+c(c-1)g^2(x)\leqq 0 \quad (a\leqq x\leqq b),$

then (P) has only the zero solution. The condition $f(x)-\theta g'(x)\leqq 0$ $(a\leqq x\leqq b)$, for some constant θ, $0\leqq\theta\leqq 1$, implies (*) for $c=\theta$. The case $\theta=0$, $\frac{1}{2}$, 1 were known to imply that (P) has only the zero solution. The proof that (*) implies this also is obtained by applying the case $\theta=0$, after the transformation

$$Y(x)=y(x)\exp\left[c\int_a^x g(t)dt\right]$$

is made in (P). *E. A. Coddington* (Copenhagen).

Dikiĭ, L. A. On the asymptotics and certain identities for the spectral function of a Sturm-Liouville operator. Dokl. Akad. Nauk SSSR (N.S.) 104 (1955), 687–690. (Russian)

The boundary-value problem $-u''+p(x)u=\lambda u$, $u'(0)=0$ on $0\leqq x<\infty$ is considered, where it is assumed that p is infinitely differentiable, the limit-point case prevails at infinity, and the spectrum is positive. Let $\theta=\theta(x, y; t)$ be the spectral function for the problem, and let M be defined by $Mf(x)=x^{-1}\int_0^{2x} f(t)dt$. Then there exist numbers r_n such that for $x, y\neq 0$,

$$M^{2k+2}[\theta(x, y; t-1)-\sum_{n=0}^{k} r_n t^{-n+\frac{1}{2}}]=O(t^{-k-\frac{1}{2}+\epsilon}),$$

as $t\to\infty$. A second result is proved, which states that

$$\lim_{T\to\infty}\int_0^T\left(1-\frac{t}{T}\right)^{2k+1}t^{k-1}[\theta(x, y; t-1)-\sum_{n=0}^{k} r_n t^{-n+\frac{1}{2}}]dt=0$$

for $x, y\neq 0$. *E. A. Coddington* (Copenhagen).

Zlámal, Miloš. Eine Bemerkung über die charakteristische Determinante einer Eigenwertaufgabe. Czechoslovak Math. J. 5(80) (1955), 175–179. (Russian summary)

Let $a_i(x)$, $b_i(x)$ be continuous functions for $a\leqq x\leqq b$; α_j^i, β_j^i polynomials in λ and $k<n$. The author considers the eigenvalue problem

(1) $\quad y^{(n)}+a_{n-1}y^{(n-1)}+\cdots+a_0y=\lambda(b_ky^{(k)}+\cdots+b_0y)$

and

(2) $\quad U_j(y)=\sum_{i=0}^{n-1}(\alpha_j^i y^{(i)}(a)+\beta_j^i y^{(i)}(b))=0 \quad (j=1, \cdots, n).$

Let $y_1(x), \cdots, y_n(x)$ be the set of fundamental solutions of (1) determined by the initial conditions $y_i^{(j)}(a)=\delta_i^{j+1}$, so that $D(\lambda)=\det(U_j(y_i))$ is the characteristic determinant of (1)–(2). By majorizing solutions of (1) by solutions of an equation with constant coefficients [Perron, Math. Ann. 113 (1937), 292–303, p. 298], it is shown that the entire function $D(\lambda)$ has an order not exceeding $1/(n-k)$. *P. Hartman* (Baltimore, Md.).

Newton, Roger G. Connection between the S-matrix and the tensor force. Phys. Rev. (2) 100 (1955), 412–428.

From the author's introduction: "In the work of Bargman, Jost and Kohn, and Levinson it was shown the scattering phase shift of one angular momentum, l, given as a function of the energy, if caused by short-range central potential, together with the L_l bound-state energies and a set of L_l positive parameters c_n, uniquely determines the potential. Moreover the latter can be constructed from the former by solving a Fredholm integral equation first derived by Gelfand and Levitan." Here the above is generalized to take account of "such noncentral potentials as that of a combination of spin-orbit coupling and the tensor force." Thus this paper goes further than the recent result of the author and Jost [Nuovo Cimento (10) 1 (1955), 590–622; MR 17, 155]. The author also generalizes a number of theorems known for central potentials concerning bound states and the low-energy behaviour of the S-matrix. *N. Levinson.*

Krumhaar, Hans. Zur Theorie der gewöhnlichen selbstadjungierten Differentialoperatoren gerader Ordnung. Math. Ann. 130 (1955), 109–136.

Let L be the formally self-adjoint ordinary differential operator of order $2m$ defined by $Lu=\sum_{\nu=0}^m(p_\nu u^{(m-\nu)})^{(m-\nu)}$. Questions of semi-boundedness connected with eigenvalue problems associated with the equation $Lu=\lambda ku$ are investigated. In addition, a rather thorough study is made of the case when the end points on the interval under consideration are regular singular points. Here the functions p_ν, k are real-valued on an open real interval $a<x<b$; $p_\nu\in C^{m-\nu-1}$, $p_\nu^{(m-\nu)}$ are piecewise continuous, $\nu=0, 1, \cdots, m-1$; p_m, k are piecewise continuous; $p_0(x)\neq 0$, $k(x)>0$ on $a<x<b$.

The first paragraph is devoted to a resumé of results due to K. Kodaira [Amer. J. Math. 72 (1950), 502–544; MR 12, 103] concerning the operator D defined by $Du=k^{-1}Lu$ for all u in an appropriate maximal domain $\mathfrak{D}$ in the Hilbert space $\mathfrak{H}$ of all functions f for which $\int_a^b|f|^2kdx<\infty$, with inner product $(f, g)=\int_a^b f\bar{g}k\,dx$. Self-adjoint real operators in $\mathfrak{H}$ associated with D are obtained by prescribing separated real self-adjoint boundary conditions at the end points of (a, b). It is shown in § 2 that an arbitrary system of boundary conditions of the above type can be replaced by an equivalent system using solutions of $(L-\lambda k)u=0$. The notions of initial numbers and normed fundamental system, which were introduced by F. Rellich in the case $2m=2$ [Math. Z. 49 (1944), 702–723; MR 7, 118] are defined, and it is shown how the boundary conditions can be expressed by linear equations in the initial numbers. The set of all systems of separated real self-adjoint boundary conditions are characterized in terms of the initial numbers. Under certain assumptions it is shown that the eigenvalues are the zeros of a determinant formed of solutions of $(L-\lambda k)u=0$.

In § 3 the operator D_0 is considered, which is the restriction of D to the set of all functions in $\mathfrak{D}$ which vanish in neighborhoods of both the end points a and b. Another proof is given of the known fact that the closure of D_0 is the adjoint of D.

It is shown in § 4 that a self-adjoint extension of D_0 is bounded from below if and only if D_0 is. In case $a<x<b$ is a finite interval, and the assumptions on p_ν, k are valid on the closed interval $a\leqq x\leqq b$ (the regular case), then D_0 is bounded from below if and only if $(-1)^m p_0(x)>0$ on $a\leqq x\leqq b$. A theorem of F. Rellich [Math. Ann. 122 (1951), 343–368; MR 13, 240] is generalized, which allows one to conclude the semi-boundedness of D_0 from the semi-boundedness at each end of the interval. This section closes with a theorem which shows a connection between m-fold zeros of solutions of $(L-\lambda k)u=0$ and the lack of boundedness from below.

In paragraphs 5 and 6 the special case is considered when a is a regular singular point of $(L-\lambda k)u=0$. The Frobenius solutions in the neighborhood of a, involving power series, powers of x, and logarithms, are introduced, and conditions on k are stated which are sufficient in order that these solutions be entire functions of λ. The characteristic exponents (roots of the indicial equation) are shown to be symmetrically situated with respect to a real point, and, except for a special case, they also are asymmetric with respect to a line parallel to the imaginary axis through this real point. These facts allow the author to prove results concerning the number of linearly independent solutions of $(L-\lambda k)u=0$ in $\mathfrak{H}$ which vanish near b. Various other results concerning the connection between the boundary conditions at a, the characteristic exponents, and semi-boundedness, are proved.

For $2m=2$ and $p_0(x)<0$, D_0 is bounded from below at a regular singular point a if and only if for one real λ all characteristic exponents are real [F. Rellich, loc. cit.]. It is shown by a counterexample that this is false for $2m>2$.

E. A. Coddington (Copenhagen).

Štraus, A. V. On spectral functions of differential operators. Izv. Akad. Nauk SSSR. Ser. Mat. 19 (1955), 201–220. (Russian)

Let $l=-DpD+q$, where $D=d/dx$, and p, q are real measurable functions on $(0, \infty)$ satisfying for all $b>0$

$$\int_0^b |p(x)|^{-1}dx<\infty, \int_0^b |q(x)|dx<\infty.$$

With an appropriate minimal domain in $L^2(0, \infty)$ this l can be viewed as a symmetric quasi-differential operator L which has deficiency index $(1, 1)$ or $(2, 2)$. The author considers the $(1, 1)$ case and gives formulas for all generalized resolutions of the identity $E(\lambda)$ associated with L. This is done by first obtaining formulas for all generalized resolvents of L [using earlier results of the author, same Izv. 18 (1954), 51–86; MR 16, 48], and then applying the Stieltjes inversion formula. The end formulas express $E(\alpha)-E(\beta)$ in terms of a fundamental set for $lu=\lambda u$ and a spectral matrix ϱ. These formulas have the same form as the known expressions for the resolutions of the identity corresponding to self-adjoint extensions of L.

E. A. Coddington (Copenhagen).

See also: Nørlund, p. 610; Skalkina, p. 631; Bellman, p. 632; Slobodyanskiĭ, p. 648; Bahvalov, p. 667; Karas, p. 687; Verde, p. 691.

Partial Differential Equations

Hornich, Hans. Über die Weiterführung eines Satzes von Peano und die Unlösbarkeit gewisser partieller Differentialgleichungen. Univ. e Politec. Torino. Rend. Sem. Mat. 14 (1954–55), 33–37.

Extending a result proved in a previous paper [Monatsh. Math. 59 (1955), 34–42; MR 16, 825], the author shows that there is a function $\phi(x, y)$, continuous in a circle B so that the equation $u_x+\phi(x, y)u_y=f(x, y)$ has no solution in any open subset G of B, whenever f is a continuous function in G which has f_y continuous and not vanishing identically in some $G'CG$. He also proves that given any operator $Lu=u_x+\psi(x, y)u_y$, where ψ is uniformly continuous in G, one can find another operator $\bar{L}u=u_x+\psi_1(x, y)u_y$ arbitrarily close — i.e. with $|\psi(x, y)-\psi_1(x, y)|<\varepsilon$ in G, for any $\varepsilon>0$ — so that the equation $\bar{L}u=f(x, y)$ is unsolvable in G for all f of the same class as in the previous theorem.

D. L. Bernstein.

Cinquini Cibrario, Maria. Nuovi teoremi di esistenza e di unicità per sistemi di equazioni a derivate parziali. Ann. Scuola Norm. Sup. Pisa (3) 9 (1955), 65–113.

Very general existence and uniqueness theorems have been given by the author (jointly with S. Cinquini) for the equation $z_x=f(x, y, z, z_y)$ with data $z(0, y)=\varphi(y)$ [Ann. Mat. Pura Appl. (4) 32 (1951), 121–155; Ann. Scuola Norm. Sup. Pisa (3) 6 (1952), 187–243; MR 13, 845; 14, 1089, in the 2nd review "q-sphere" should read "q-space"]. Similar theorems are now established for the system $p_i=f_i(x, y, z_k, q_i)$ $(i, k=1, \cdots, m;\ p_i\equiv z_{ix},\ q_i\equiv z_{iy})$ in a strip $D_a(-\infty<y<\infty,\ 0\leq x\leq a$, various $a>0)$ with data $z_i(0, y)=\varphi_i(y)$. First the quasi-linear system (Q) $p_i+\varrho_i(x, y, z_k)q_i=f_i(x, y, z_k)$ is discussed with ϱ_i and f_i quasi-continuous in x and continuous in (y, z_k), and φ_i Lipschitzian. The argument uses Ascoli's theorem and depends on certain integral equations introduced under stronger hypotheses by Courant and Lax [Comm. Pure Appl. Math. 2 (1949), 255–273; MR 11, 441]. Functions $z_i(x, y)$ are obtained that are absolutely continuous in x and Lipchitzian in y and satisfy (Q) almost everywhere in D_a. It is next shown that certain smoothness conditions on ϱ_i, f_i, and φ_i imply that q_i are absolutely continuous in x and Lipschitzian in y. Then the results are extended to nonlinear systems of the form $p_i=f_i(x, y, z_k, q_i)$, for which uniqueness theorems had already been given by S. Cinquini [Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) 17 (1954) 188–191, 339–344; MR 16, 1113], and an extension is indicated to systems of the form

$$F_h(x, y; z_i; p_i; q_k)=0.$$

Finally, it is shown that for each existence theorem a corresponding uniqueness theorem can be obtained, and that z_i depend continuously on the data φ_i.

F. A. Ficken (Knoxville, Tenn.).

Baiada, E.; e Vinti, C. Un teorema d'esistenza della soluzione per un'equazione alle derivate parziali del 1° ordine. Ann. Scuola Norm. Sup. Pisa (3) 9 (1955), 115–160.

An earlier paper by Baiada [Ann. Mat. Pura Appl. (4) 34 (1953), 1–25; MR 14, 1089] presented an existence theorem for the equation $z_x=f(x, z_y)$. Using very similar methods the authors here obtain a very similar theorem for the equation $z_x=f(x, y, z, z_y)$. [A single estimate covers two pages (140–1).]

F. A. Ficken.

Koševel, A. I. Differentially corresponding spaces and existence theorems. Dokl. Akad. Nauk SSSR (N.S.) 105 (1955), 22–25. (Russian)

Let Ω be a bounded region in n-dimensional Euclidean space. A pair (X, Y) of Banach spaces of real-valued functions on Ω is called a differentially corresponding pair if (1) the second partial derivatives together with the identity are continuous operators from X to Y and the first derivatives are continuous from X to both Y and C, the space of continuous functions; and (2) multiplication is continuous from $(Y\cap C)\times Y$ to Y. In particular, such a pair is $(W_p^{(2)}(\Omega), L_p(\Omega))$ $(p>n)$, where $W_p^{(2)}$ is the space of functions whose second derivatives exist in a generalized sense and belong to L_p. The author is concerned with solving second-order quasi-linear differential equations of elliptic type, $P(u)=0$, with $u=0$ on the boundary of Ω. If the coefficients have continuous second derivatives, then

$P(u)$ as a map from $W_p^{(2)}$ to L_p is continuous and has a second Fréchet differential. Furthermore, if the first Fréchet differential has a bounded inverse, then he can rely on Kantorovič's formulation [Uspehi Mat. Nauk (N.S.) 3 (1948), no. 6(28), 89–185; MR 10, 380] of "Newton's method". Several theorems using these concepts are announced.
G. Hufford (Stanford, Calif.).

Bagirov, H. G. On the correctness of formulation of Goursat's problem for a system. Akad. Nauk Azerbaĭdžan. SSR. Trudy Inst. Fiz. Mat. 4-5 (1952), 31–37. (Russian. Azerbaijani summary)

Consider the system

$$(*)\quad \frac{\partial}{\partial x_1}u_{i_1}=\Phi_{1i_1}(x_1, \cdots, x_s; u_1, \cdots, u_N)\ (i_1=1, \cdots, n_1)$$

$$\frac{\partial}{\partial x_2}u_{i_2}=\Phi_{2i_2}(x_1, \cdots, x_s; u_1, \cdots, u_N)\quad (i_2=n_1+1, n_1+2, \cdots, n_1+n_2)$$

$$\cdots$$

$$\frac{\partial}{\partial x_s}u_{i_s}=\Phi_{si_s}(x_1, \cdots, x_s; u_1, \cdots, u_N)\quad (i_s=n_1+n_2+\cdots+n_{s-1}+1, \cdots, n_1+n_2+\cdots+n_s=N)$$

with the initial conditions

$$u_{i_1}|_{x_1=x_1^0}=\phi_{i_1}(x_2, x_3, \cdots, x_s)\ (i_1=1, 2, \cdots, n_1),$$
$$u_{i_2}|_{x_2=x_2^0}=\phi_{i_2}(x_1, x_3, \cdots, x_s)\ (i_2=n_1+1, \cdots, n_1+n_2),$$
$$\cdots$$
$$u_{i_s}|_{x_s=x_s^0}=\phi_{i_s}(x_1, x_2, \cdots, x_{s-1})\quad (i_s=n_1+n_2+\cdots+n_{s-1}+1, \cdots, N).$$

This is called the Goursat problem for the system (*). It is "correctly set in the sense of Hadamard" if the solutions exist, are unique, and depend continuously on the initial data.

The question whether the Goursat problem is correctly set for the system (*) was raised by Petrovsky [Uspehi Mat. Nauk (N.S.) 1 (1946), no. 3-4(13-14), 44–70; MR 10, 301; 11, 520]. It is answered in the affirmative, under suitable hypotheses, in the present paper. Perhaps the essential hypothesis is a Lipschitz condition on the Φ_{ji_s}. The proof involves rewriting (*) immediately as a system of non-linear integral equations, which are then solved by successive approximations. Convergence of the successive approximations follows directly from the Lipschitz condition on the Φ_{ji_s}.
R. B. Davis (Durham, N.C.).

Finzi, A. On a conjecture of M. Janet. Technion. Israel Inst. Tech. Sci. Publ. 6 (1954/5), 34–37. (Hebrew summary)

The author ascribes to Janet the following conjecture: If the solution of a system of n partial differential equations in n unknown functions of $m+1$ independent variables does not depend on any arbitrary function of m variables, then it does not depend on any arbitrary function of fewer than n variables. Without giving precise meaning to this conjecture, the author sketches a procedure which, if capable of being carried out, will, in his opinion, settle the question.
E. R. Kolchin.

Cilberto, Carlo. Il problema di Darboux per una equazione di tipo iperbolico in due variabili. Ricerche Mat. 4 (1955), 15–29.

The author gives a new proof, based upon a fixed-point of the result established by Hartman and Wintner: if $f(x, y, z, p, q)$ is continuous, bounded, and satisfies a uniform Lipschitz condition with respect to p and q in a region S: $0\leq x\leq a$; $0\leq y\leq b$; $-\infty<z, p, q<+\infty$, and if $\sigma(x)$ and $c(y)$ are functions of class C^1 with $\sigma(0)=c(0)$, then there exists a solution $z(x, y)$ of

$$(*)\quad z_{xy}=f(x, y, z, z_x, z_y);\ z(x, 0)=\sigma(x);\ z(0, y)=c(y),$$

in the rectangle R: $0\leq x\leq a$; $0\leq y\leq b$. He then establishes, again by means of a fixed-point theorem, another result concerning the existence of a solution of (*) when $\sigma(x)=c(y)=0$, under hypotheses of a different nature. They are too long to be given here in detail, but they are somewhat analogous to those of the Osgood uniqueness theorem for ordinary differential equations, involving the existence of functions $g(x, y)$ and $\omega(u)$ such that

$$|f(x, y, z, p_2, q_2)-f(x, y, z, p_1, q_1)|\leq g(x, y)\cdot \omega[(p_2-p_1)+(q_2-q_1)].$$

Preliminary to proving this result, he proves the equicontinuity of a set of functions $v(\lambda, t)$, which are each solutions of an ordinary differential equation in t, for various parameter functions. This preliminary result (theorem II) can probably be used to establish existence theorems for other partial differential equations.
D. L. Bernstein (Rochester, N.Y.).

★ **Springer, George. Baecklund transformations which leave partial differential equations invariant.** Nonlinear differential equations of the second order, pp. 73–78. OOR Project No. 956, technical report. Northwestern University, Evanston, Ill., 1955.

On order

A system of first-order partial differential equations

$$(*)\quad \xi_u=H\xi_v\ \left(\xi=\binom{\xi}{\eta},\ H=\binom{h_{11}h_{12}}{h_{21}h_{22}}\right)$$

can be associated to a system $\xi_u'=H'\xi_v'$ by a Baecklund transformation whose coefficients are 2×2 matrices depending upon u and v and satisfying a certain system of five equations [cf. Loewner, J. Analyse Math. 2 (1953), 219–242; MR 15, 225]. If the 2×2 matrices depend upon u, v and t, the notion of an infinitesimal Baecklund transformation can be introduced. Then any twice continuously differentiable solution of

$$(**)\quad \xi_u=H(u, v, t)\xi_v$$

for a particular value t' of t can be extended to a function $\xi(u, v, t)$ satisfying (**) for all values of t and one may even prescribe $\xi(u_0, v_0, t)$ for a point (u_0, v_0) contained in a region Δ in the (u, v)-plane arbitrarily as a once continuously differentiable function of t. The author poses the problem of finding those Baecklund transformations of the form

$$(***)\quad \begin{matrix}\xi_u'=W\xi_u+A\xi+C\xi'\\ \xi_v'=W\xi_v+B\xi+D\xi'\end{matrix}\ \text{or}\ \begin{matrix}\xi_{ut}=\Omega\xi_u+A\xi+C\xi_t\\ \xi_{vt}=\Omega\xi_v+B\xi+D\xi_t\end{matrix}$$

which transform a given equation into itself. In this way, from one solution of (*) or (**) the author may generate a solution involving arbitrary parameters. Then new solutions may be found in terms of a given solution by solving (***). Then the matrices W and H or Ω and H must commute. In the case where D commutes with H the differential equations

$$(****)\quad \xi_u=h_1\eta_v,\ \eta_v=\epsilon k^2h_1\xi_v$$

are transformed into itself. The method introduces from one solution a new solution with six arbitrary real parameters. The equations (****) become the Laplace equation

(Cauchy-Riemann equations) when $h_1\equiv 1$, $k=1$ and $\varepsilon=-1$, and the wave equation when $h_1\equiv 1$, $k=1$ and $\varepsilon=+1$. In general, when $\varepsilon=-1$ the system is elliptic, when $\varepsilon=+1$ the system is hyperbolic. In a more general case the author assumes that $B=cD$, c=constant, and drops the assumption that D commutes with H. *M. Pinl.*

★ **Scott, W. T. Linear Baecklund transformations.** Non-linear differential equations of the second order, pp. 79–92. OOR Project No. 956, technical report. Northwestern University, Evanston, Ill., 1955.

In general, for $m\geqq 1$ and $n\geqq 2$ the analytic definition of a Baecklund transformation for real-valued functions of real variables is given by a system of $2m+n$ relations between functions $y^i(x^1, \cdots, x^n)$, $\eta^i(\xi^1, \cdots, \xi^n)$ $(i=1, 2, \cdots, n)$ and the first derivative of these functions. For simplicity the author suppresses the effect of a point transformation $(x^1, x^2, \cdots, x^n)\to(\xi^1, \xi^2, \cdots, \xi^n)$ by putting $\xi^i=x^i$ $(i=1, 2, \cdots, n)$ and considering a system of $2m$ relations between the functions $y^i(x^1, x^2, \cdots, x^n)$, $\eta^i(x^1, x^2, \cdots, x^n)$ $(i=1, 2, \cdots, m)$ and their first partial derivatives ((m, n)-transformation). A linear (m, n)-transformation of the form

$$(*)\quad \eta_i=A^{i0}y+\sum_{j=1}^{n} A^{ij}y_j+B^{i0}\eta+\sum_{\nu=1}^{n} B^{ij}\eta_j+C^i \quad (i=1, 2),$$

(matrix notation), is said to be reversible if it can be written in the form

$$(**)\quad y_i=\alpha^{i0}\eta+\sum_{j=1}^{n} \alpha^{ij}\eta_j+\beta^{i0}y+\sum_{j=3}^{n} \beta^{ij}y_j+\gamma^i \quad (i=1, 2).$$

The (m, n)-transformation is reversible if and only if there exist $m\times m$ matrices α^{ij} $(i, j=1, 2)$, for which

$$A^{i1}\alpha^{1j}+A^{i2}\alpha^{2j}=\delta^{ij}I,\ \alpha^{i1}A^{1j}+\alpha^{i2}A^{2j}=\delta^{ij}I\ (i, j=1, 2).$$

By putting schematically

$$M=\begin{pmatrix}A^{11}A^{12}\\A^{21}A^{22}\end{pmatrix}$$

the following theorem holds: The linear (m, n)-transformation (*) is reversible if and only if the $2m\times 2m$ matrix M is non-singular. In a special section the author determines sufficient conditions on the coefficients of a linear (m, n)-transformation (*) in order that the functions y appearing in (*) shall satisfy a system of partial differential equations. Following the method used by Loewner [NACA Tech. Note no. **2065** (1950); J. Analyse Math. **2** (1953), 219–242; MR **13**, 464; **15**, 225] and by Resch [Thesis, Syracuse, 1951] conditions are determined under which η and its partial derivatives can be eliminated between (*) and the relations obtained from (*) by one partial differentiation. The result of such an elimination will be a system of second- or first-order linear partial differential equations in y which will be satisfied by y provided y and η are linked by the (m, n)-transformation. If the linear (m, n)-transformation (*) is reversible, similar considerations for (**) lead to a system of partial differential equations in η; in this case the systems of partial differential equations in y and in η are linked by the reversible linear (m, n)-transformation (*) or (**). Under which conditions a second-order system

$$(***)\qquad \sum_{j,k=1}^{n} R^{jk}y_{jk}+\sum_{j=1}^{n} S^jy_j+S^0y+T=0$$

of the elimination mentioned above can be reduced to a first-order system? The coefficients R^{ik} of (***) satisfy

$$R^{11}=R^{22}=0,\ R^{12}+R^{21}=0,\ R^{ik}+R^{ki}=0$$
$$(i=1, 2;\ k=3, 4, \cdots, n),$$

if and only if

$$A^{12}=A^{21}=0,\ A^{11}=A^{22}=A,\ A^{ik}=-B^{ik}A$$
$$(i=1, 2;\ k=3, 4, \cdots, n)$$

(then the matrix A is non-singular). Linked first-order systems have the same characteristic equation. Finally the special cases $n=2$ and $m=1$ are discussed.

M. Pinl (Cologne).

★ **Resch, Daniel. Some Baecklund transformations of partial differential equations of second order.** Non-linear differential equations of the second order, pp. 97–114. OOR Project No. 956, technical report. Northwestern University, Evanston, Ill., 1955.

C. Loewner [NACA Tech. Note no. **2065** (1950); MR **13**, 464] used Baecklund transformations to link the system of linear first-order partial differential equations

$$a_1{}^iu_x+b_1{}^iu_y+a_2{}^iv_x+b_2{}^iv_y=0,$$
$$a_1{}^i=a_1{}^i(x, y),\ \cdots,\ b_2{}^i=b_2{}^i(x, y)\quad (i=1, 2),$$

representing the steady two-dimensional flow of a compressible fluid (in the hodograph plane) to a simpler system of equations. The problem is to find in the four-dimensional spaces (x, y, u, v) and (x', y', u', v') all two-dimensional surfaces Σ: $u=u(x, y)$, $v=v(x, y)$ and Σ': $u'=u'(x', y')$, $v'=v'(x',y')$ and all one-to-one mappings between Σ and Σ' such that at corresponding points six given equations

$$G_i(x, y, u, v, u_x, u_y, v_x, v_y;\ x', y', u', v', u_x', u_y', v_x', v_y')=0$$
$$(i=1, 2, \cdots, 6),$$

are satisfied. The author attempts to find the proper Baecklund transformation which links second-order partial differential equations in three independent variables and admits only those Baecklund transformations which are linear in the quantities z, $\bar{z}$ and their first derivatives. If the assumption is made, that three of the five equations

$$(*)\ F_i\,(x_1, x_2, x_3, z, p_1, p_2, p_3;\ x_1', x_2', x_3', z', p_1', p_2', p_3')=0$$
$$(i=1, 2, 3, 4, 5)$$

represent the relation between the ground spaces, the theory can be carried through for the particular Baecklund transformation $(i=1, 2;\ j=0, 1, 2, 3)$

$$(**)\ \alpha^{11}z_{x_1}+\alpha^{12}z_{x_2}+\alpha^{13}z_{x_3}-\alpha^{10}z+\beta^{11}\bar{z}_{x_1}+\beta^{12}\bar{z}_{x_2}+\beta^{13}\bar{z}_{x_3}-\beta^{10}\bar{z}=0,\ \alpha^{ij}=\alpha^{ij}(x_1,x_2,x_3),$$

$$\alpha^{21}z_{x_1}+\alpha^{22}z_{x_2}+\alpha^{23}z_{x_3}-\alpha^{20}z+\beta^{21}\bar{z}_{x_1}+\beta^{22}\bar{z}_{x_2}+\beta^{23}\bar{z}_{x_3}-\beta^{20}\bar{z}=0,\ \beta^{ij}=\beta^{ij}(x_1,x_2,x_3),$$

or

$$A^iz+B^i\bar{z}=0\ (=i1, 2),$$

if

$$A^iu=\alpha^{i1}u_{x_1}+\alpha^{i2}u_{x_2}+\alpha^{i3}u_{x_3}-\alpha^{i0}u,$$
$$B^iu=\beta^{i1}u_{x_1}+\beta^{i2}u_{x_2}+\beta^{i3}u_{x_3}-\beta^{i0}u\quad (i=1, 2).$$

Differentiating (**) with respect to x_1, x_2 and eliminating z, the author gets under some restrictive conditions a partial differential equation of the second order in $\bar{z}$. If the roles of z and $\bar{z}$ are interchanged similarly, a partial differential equation in z can be obtained. The equations

(**) form a complete system in z, if

$$\begin{vmatrix} \alpha^{11} & \alpha^{12} & \alpha^{13} \\ \alpha^{21} & \alpha^{22} & \alpha^{23} \\ [\alpha^1\alpha^2]^1 & [\alpha^1\alpha^2]^2 & [\alpha^1\alpha^2]^3 \end{vmatrix} = \begin{vmatrix} \alpha^{11} & \alpha^{12} & \alpha^{10} \\ \alpha^{21} & \alpha^{22} & \alpha^{20} \\ [\alpha^1\alpha^2]^1 & [\alpha^1\alpha^2]^2 & \alpha^0 \end{vmatrix} + \begin{vmatrix} \alpha^{11} & \alpha^{13} & \alpha^{10} \\ \alpha^{21} & \alpha^{23} & \alpha^{20} \\ [\alpha^1\alpha^2]^1 & [\alpha^1\alpha^2]^3 & \alpha^0 \end{vmatrix} + \begin{vmatrix} \alpha^{12} & \alpha^{13} & \alpha^{10} \\ \alpha^{22} & \alpha^{23} & \alpha^{20} \\ [\alpha^1\alpha^2]^2 & [\alpha^1\alpha^2]^3 & \alpha^0 \end{vmatrix} = 0,$$

$[\alpha^1\alpha^2]^i=\alpha^{1j}\alpha_{x_j}{}^{2i}-\alpha^{2j}\alpha_{x_j}{}^{1i}$; $\alpha^0=\alpha^{1j}\alpha_{x_j}{}^{20}-\alpha^{2j}\alpha_{x_j}{}^{10}$; $i,j=1, 2, 3,$ and

$$(***)\quad \begin{vmatrix} \alpha^{11} & \alpha^{12} & B^1\bar z \\ \alpha^{21} & \alpha^{22} & B^2\bar z \\ [\alpha^1\alpha^2]^1 & [\alpha^1\alpha^2]^2 & B(\bar z) \end{vmatrix} + \begin{vmatrix} \alpha^{11} & \alpha^{13} & B^1\bar z \\ \alpha^{21} & \alpha^{23} & B^2\bar z \\ [\alpha^1\alpha^2]^1 & [\alpha^1\alpha^2]^3 & B(\bar z) \end{vmatrix} + \begin{vmatrix} \alpha^{12} & \alpha^{13} & B^1\bar z \\ \alpha^{22} & \alpha^{23} & B^2\bar z \\ [\alpha^1\alpha^2]^2 & [\alpha^1\alpha^2]^3 & B(\bar z) \end{vmatrix} = 0,$$

$$B(\bar z)=\alpha^{1j}(B^2\bar z)_{x_j}-\alpha^{2j}(B^1\bar z)_{x_j}+\alpha^{20}B^1\bar z-\alpha^{10}B^2\bar z.$$

The second condition (***) represents a partial differential equation in $\bar z$. The Baecklund transformation can be applied to transform a general second-order partial differential equation into one of the same type but with different terms of first and zeroth order. The author considers the example

$$\alpha^{11}=\alpha^{22}=1,\ \alpha^{12}=\alpha^{21}=\alpha^{13}=\alpha^{23}=0.$$

The result is: The Baecklund transformation

$$z_{x_1}+\frac{1}{x_1-x_2}z+\frac{x_2-x_1}{2x_3}\bar z_{x_1}-\bar z_{x_2}-\frac{x_1-x_2-x_3}{(x_1-x_2)x_3}\bar z=0$$

$$z_{x_2}+\frac{1}{x_2-x_1}\bar z+\bar z_{x_1}-\frac{x_2-x_1}{2x_3}\bar z_{x_2}+\bar z_{x_3}+\frac{x_1-x_2-x_3}{(x_1-x_2)x_3}\bar z=0$$

links the equation

$$z_{x_1x_1}+z_{x_2x_2}+z_{x_1x_3}+z_{x_2x_3}+\frac{1}{x_3}\left(z_{x_1}+z_{x_2}\right)-\frac{4}{(x_1-x_2)^2}z=0$$

with equation

$$\bar z_{x_1x_1}+\bar z_{x_2x_2}+\bar z_{x_1x_3}+\bar z_{x_2x_3}=0.$$

It can be showed that these differential equations otherwise could not be connected by a transformation of the independent or dependent variables. The particular Baecklund transformation (*) can only link partial differential equations of hyperbolic type (if not degenerate).

M. Pinl (Cologne).

★ **Weinstein, A. The method of axial symmetry in partial differential equations.** Convegno Internazionale sulle Equazioni Lineari alle Derivate Parziali, Trieste, 1954, pp. 86–96. Edizioni Cremonese, Roma, 1955. 3000 Lire.

This paper is a review of two problems regarding the Euler-Poisson-Darboux equation, the Cauchy problem and the radiation problem. The Euler-Poisson-Darboux equation

$$\frac{\partial^2 u}{\partial y^2}+\frac{k}{y}\frac{\partial u}{\partial y}=\Delta u,\quad -\infty<k<+\infty,$$

$$u=u(x_1, \cdots, x_m),\ \Delta u=\sum_{i=1}^{m}\frac{\partial^2 u}{\partial x_i{}^2},$$

can be solved for all values of k and m by typical formulas according to whether k is equal to, greater than, or less than $m-1$. The Cauchy problem consists in finding a solution satisfying the initial conditions $u^{(k)}(x, 0)=f(x)$, $u_y{}^{(k)}(x, 0)=0$, where $x=(x_1, \cdots, x_m)$, for all values of m and k.

I. If $k=m-1$, the solution, as given explicitly by Asgeirsson, is

$$u^{(m-1)}(x, y)=\frac{1}{\omega_m}\int\cdots\int_{\sum_1^m\alpha_i^2=1} f(x_1+\alpha_1 y, \cdots, x_m+\alpha_m y)\,d\omega_m = M(x, y, f),$$

where $\omega_m=2\pi^{m/2}\Gamma(m/2)$ and M denotes the mean value of f [Math. Ann. **113** (1936), 321–346].

II. For $k>m-1$ the author obtained the formula

$$u^{(k)}(x, y)=\frac{\omega_{k+1-m}}{\omega_{k+1}}\int\cdots\int_{\sum_1^m\alpha_i^2\leq 1} f(x_1+\alpha_1 y, \cdots, x_m+\alpha_m y)\cdot (1-\sum\alpha_i{}^2)^{(k-m-1)/2}d\alpha_1 d\alpha_2\cdots d\alpha_m.$$

A new proof of this formula was recently given by F. Bureau [Comm. Pure Appl. Math. **8** (1955), 143–202; MR **16**, 826]. By introducing polar coordinates Diaz and Weinstein have written the solution in the form

$$u^{(k)}(x, y)=\frac{2\Gamma((k+1)/2)}{\Gamma((k+1-m)/2)\Gamma(m/2)}\int_0^1 M(x, \varrho y; f)\cdot(1-\varrho^2)^{(k-m-1)/2}\varrho^{m-1}d\varrho,$$

where $M(x, \varrho y, f)$ is again a mean value.

III. For $k<m-1$ but $k\neq -1, -3, -5, \cdots$ two formulas were found by the author by using recursion formulas. In particular, the solution

$$u^{(k)}=y^{1-k}\left(\frac{\partial}{y\partial y}\right)^n\left(y^{k+2n-1}u^{(k+2n)}\right)$$

includes as a particular case the formula of Poisson for the three-dimensional wave equation and the formulas of Poisson-Parseval-Volterra for cylindrical waves. A solution of the Cauchy problem for arbitrary initial values in the exceptional cases $k=-1, -3, -5, \cdots$ was given by Diaz and Weinberger [Proc. Amer. Math. Soc. **4** (1953), 703–715; MR **15**, 321] and soon after by Blum [ibid. **5** (1954), 511–520; MR **16**, 137]. It is sufficient to assume that the given function has derivatives of order at least $(m-k+3)/2$.

The generalized radiation problem can be formulated as following. Find a solution of the Euler-Poisson-Darboux equation

$$u_{xx}-u_{yy}-\frac{k}{y}u_y=0$$

which satisfies the boundary conditions

$$u(x, 0)=f(x),\ u(x, y)=0$$

on the characteristic $y=x$. The problem covers the radiation problem for $k=3-m$ and the Tricomi problem for $k=\frac{1}{3}$. Its solution is given by

$$u^{(k)}(x, y)=\frac{1}{\Gamma(1-k)}\int_0^{x-y} I^{n+k}f^{(n+1)}(\xi)[(x-\xi)^2-y^2]^{-k/2}d\xi+\frac{1}{\Gamma(n+k+1)}\sum_{h>k/2}^{n} f^{(h)}(0)\frac{\partial^{n-h+1}A_{n+k}(x, y)}{\partial x^{n-h+1}},$$

where

$$A_\alpha(x, y)=\frac{1}{\Gamma(1-k)}\int_0^{x-y}\xi^\alpha[(x-\xi)^2-y^2]^{-k/2}d\xi,\quad \alpha>-1,$$

$$k=-n+\beta,\ n=n(k)=0, 1, 2, \cdots; 0\leq\beta<1,$$

and I^{n+k} denotes the Riemann-Liouville integral.

M. Pinl (Cologne).

Chen, Y. W. Degenerate solutions of partial differential equations. Proc. Amer. Math. Soc. 6 (1955), 855–861.

The equation of motion of compressible fluids is of the form

$$(*)\quad L(u)=\sum_{i,j=1}^{n} a_{ij}(p_1, p_2, \cdots, p_n)\partial^2u/\partial x_i\partial x_j=0 \qquad \left(p_i=\frac{\partial u}{\partial x_i}\right).$$

A solution of (*), whose first partial derivatives satisfy $n-s$ functional relations

$$p_\alpha=F^\alpha(p_1, \cdots, p_s)\ (\alpha=s+1, s+2, \cdots, n)$$

among themselves is called a "s-tuple wave" (degenerate compressible flow). A necessary and sufficient condition for the existence of a degenerate solution u, of the type of an s-tuple wave, with nonvanishing determinant $|\partial^2u/\partial x_k\partial x_i|$ is the following: the $n-s+1$ functions F^α and G satisfy a system of $C(n-1, s-1)$ differential equations of second order and degree $s-1$, in the variables p_k,

$$(**)\quad (a_{ik}+2a_{i\beta}\partial F^\beta/\partial p_k+a_{\alpha\beta}\partial F^\alpha/\partial p_i\partial F^\beta/\partial p_k)P_{ik}^{(m)}=0$$

with

$$|x_\alpha\partial^2F^\alpha/\partial p_i\partial p_k+\partial^2G/\partial p_i\partial p_k|\neq0.$$

The author introduces new independent variables $x_1', \cdots, x_n'$ and dependent variable $u'(x_1', \cdots, x_n')$ by the elementary contact transformation:

$$x_k'=p_k,\ x_\alpha'=x_\alpha,\ u'=u-p_kx_k,\ p_k'=-x_k,\ p_\alpha'=p_\alpha.$$

Hence one obtains:

$$G(x_1', x_2', \cdots, x_s')=u'-x_\alpha'F^\alpha(x_1', \cdots, x_s').$$

Each $P_{ik}^{(m)}$ is a homogeneous polynomial of degree $s-1$ in the second derivatives of F^α and G with integer coefficients. In the case of double waves ($s=2$) there are $n-1$ equations and the same number of unknown functions $F^\alpha(p_1, p_2)$ and $G(p_1, p_2)$. Centered waves are those with $G=0$. An interesting example is given by $L(u)=\Delta^2u$ (x_1, x_2, x_3). The corresponding equation (**) is the differential equation of minimal surfaces. The solution of the initial-value problem for the functions $F^\alpha(p_1, p_2)$ ($s=2$, problem I), which exists uniquely in a neighborhood of the initial curve Γ in the (p_1, p_2)-plane:

$$\Gamma: p_1=s_1(\tau),\ p_2=s_2(\tau)$$

gives a solution of the initial-value problem for the unknown function $u(x)$ (in the neighborhood of an initial manifold M_{n-1} in the x-space, problem II). Conversely, one can find by simple means of differentiation and elimination from an unique solution $u=U(x)$ of problem II the solution of problem I. The author discusses especially the case of constant coefficients a_{ij}, $L(u)$ is totally and the initial manifold is "space like". *M. Pinl.*

Schaefer, Helmut. Eine Bemerkung über hyperbolische Systeme partieller Differentialgleichungen zweiter Ordnung. Jber. Deutsch. Math. Verein. 58 (1955), Abt. 1, 39–42.

The hyperbolic equation of second order in canonical form

$$(*)\quad u_{xy}=f(x, y, u, u_x, u_y)$$

has as characteristics the lines x=const. and y=const. The two standard problems for this equation are the Cauchy problem in which a solution is required which assumes given values on a non-characteristic curve, and the Goursat problem in which a solution is required which assumes given values on two intersecting characteristic lines. In all the solutions given for these problems, under various hypotheses regarding f and the initial data, the solution obtained is shown to have continuous derivatives u_x, u_y, u_{xy}. Nevertheless, it is often important to know that the other two second-order derivatives u_{xx} and u_{yy} also exist and are continuous; for example, these conditions are required to reduce the general hyperbolic equation to the form (*). In the present paper the author shows that this is true for the Goursat problem, and that the approximating solutions have all first and second order derivatives converging uniformly to the corresponding derivatives of u. The Cauchy problem can be handled in an analogous fashion and indeed the extension to systems of equations of the form

$$u_{i,xy}=f_i(x, y, u_j, p_j, q_j)\ (i, j=1, 2, \cdots, m)$$

can easily be done. His basic hypothesis on f is that it be of class C^1. *D. L. Bernstein* (Rochester, N.Y.).

Dacev, A. B. On the two-dimensional Stefan problem. Dokl. Akad. Nauk SSSR (N.S.) 101 (1955), 441–444. (Russian)

Dacev, A. B. On the three-dimensional problem of Stefan. Dokl. Akad. Nauk SSSR (N.S.) 101 (1955), 629–632. (Russian)

These two papers consider the problem of Stefan in two- and three-dimensions respectively and are similar in method. As a model of the type of problem considered one might think of the temperature distribution problem encountered in the melting of a piece of ice completely submerged in water, the water and ice representing the two states.

If $u^{(i)}(x, y, z, t)$ $(i=1, 2)$ represents the temperature distributions in the regions R_1 (water) and R_2 (ice), which are separated by a smooth surface C, then $u^{(i)}$ must satisfy the equations

$$(1)\quad a_i^2\nabla^2u^{(i)}=\partial u^{(i)}/\partial t$$

and the initial conditions

$$(2)\quad u^{(i)}(x, y, z, t_0)=\Phi_i(x, y, z)$$

and the boundary conditions

$$(3)\quad u^{(i)}(C)=0.$$

If d is a member of the family of curves which are the orthogonal trajectories of C and s is the arc length measured along d inward from the initial surface C_0 then a point $M(x, y, z)$ on the surface C may be located parametrically by the equations

$$(4)\quad x=f(\alpha, \beta, s(\alpha, \beta, t)),\ y=g(\alpha, \beta, s(\alpha, \beta, t)), \quad z=h(\alpha, \beta, s(\alpha, \beta, t)),$$

where the functions f, g, h satisfy the conditions

$$(4')\quad f_t^2+g_t^2+h_t^2=s_t^2,\ f_tf_\alpha+g_tg_\alpha+h_th_\alpha=0, \quad f_tf_\beta+g_tg_\beta+h_th_\beta=0.$$

The condition of Stefan requires that

$$(5)\quad ds/dt=\varepsilon(k_1\partial u^{(1)}/\partial\nu-k_2\partial u^{(2)}/\partial\nu)$$

for (x, y, z) on C and ν the interior normal to C.

The problem requires that functions $u^{(1)}$, $u^{(2)}$, s, f, g, h be found which satisfy (1), (2), (3), (4') and (5).

An auxiliary problem is first considered, which requires

that, given (4), a function $u(x, y, z, t)$ be found which satisfies the heat equation (1) in a bounded region R, the initial conditions (2) and the boundary conditions (3). This is done by breaking the region R into layers and solving the problem in each layer and matching these solutions at the surfaces separating the layers. A solution in each layer is found by converting the problem into an integral equation and finding a series solution of it. This procedure gives solutions $u_{in}^{(1)}$, $u_{in}^{(2)}$ in $R_{in}^{(1)}$ and $R_{in}^{(2)}$ which satisfy (1), (2) and (3).

Now let t be such that $t_{i-1}<t\leqq t_i$. The condition (5) permits the computation of

$$(6)\qquad s_{in}(\alpha, \beta, t)=s_{i-1,n}(\alpha, \beta, t)+ e\int_{t_{i-1}}^{t}\left(k_1\frac{\partial u_{in}^{(1)}}{\partial \nu_i}-k_2\frac{\partial u_{in}^{(2)}}{\partial \nu_i}\right)dt,$$

where for purposes of integration (x, y, z) is taken as a point on the surface C_{i-1}. Since (x_0, y_0, z_0) is known, this permits sequential determination of the s_{in}, f_{in}, g_{in}, h_{in}. It is asserted that the approximating functions $u_{in}^{(1)}$, $u_{in}^{(2)}$, s_{in}, f_{in}, g_{in}, h_{in} converge to the solution of the problem as n approaches infinity. *C. G. Maple.*

Dacev, A. B. On the three-dimensional multilayer problem of heat conduction. Dokl. Akad. Nauk SSSR (N.S.) **101** (1955), 1019–1021. (Russian)

Dacev, A. B. On the two-dimensional multilayer problem of heat conduction. Dokl. Akad. Nauk SSSR (N.S.) **101** (1955), 813–816. (Russian)

These two papers consider the heat conduction problem for two media which are in contact along a surface. The problems and methods of solution are the same except for minor details. For the three-dimensional case it may be stated as follows: Let S be a closed bounded surface separating the infinite region R_2 from the bounded region R_1 which are occupied by homogeneous media A_2 and A_1. If $u_i(x, y, z, t)$ $(i=1, 2)$ denotes the temperature in the region R_i, then u_i must satisfy: 1) $a_i^2\Delta u_i=\partial u_i/\partial t$; the initial conditions 2) $u_i(r, 0)=\Phi_i(r)$; and the continuity conditions on S, 3) $u_1=u_2$, 3′) $k_1\partial u_1/\partial \nu=k_2\partial u_2/\partial \nu$, where ν is the interior normal to S. The solution functions u_i are expressed in the form 4) $u_i(r, t)=V_i(r, t)+W_i(r, t)$, where the V_i are expressed directly in terms of the initial functions. The W_i are given in terms of functions μ_i which are solutions of a certain integral equation which in turn depends upon an unknown function $f(r, t)$. The functions u_1 and u_2 so determined satisfy condition (3) automatically and f is determined through condition (3′).
C. G. Maple (Ames, Ia.).

Rubinšteĭn, L. I. On determination of the boundary separating phases in a two-phase heat-conducting medium in a steady heat regime. Dokl. Akad. Nauk SSSR (N.S.) **105** (1955), 437–438. (Russian)

Let D be a doubly connected region in E^3 bounded by the closed surfaces Σ_0 and Σ_1. Consider the problem: find a surface Σ and functions $U_i(P)(i=0, 1)$ such that if

$$(1)\qquad \overline{D_0+D_1}=\bar{D},\ D_0\cdot D_1=0,\ \bar{D}_i=\Sigma_i+\Sigma+D_i,$$

then $U_i(P)$ is harmonic inside D_i and satisfies the conditions

$$(2)\qquad \begin{aligned} &U_0(P)=f_0(P)<0, && P\in\Sigma_0,\\ &U_1(P)=f_1(P)>0, && P\in\Sigma_1,\\ &U_0(P)=U_1(P)=0, && P\in\Sigma,\\ &\lambda_0\frac{\partial U_0}{\partial N}=\lambda_1\frac{\partial U_1}{\partial N}, && P\in\Sigma, \end{aligned}$$

where N is the interior normal relative to D_1.

The substitutions

$$(3)\qquad v_i(P)=\lambda_i U_i(P),\ \phi_i(P)=\lambda_i f_i(P)\ (i=0, 1)$$

reduce the problem to:

$$(4)\qquad \Delta v_i=0,\ P\in D_1;\ v_0=\phi_0<0,\ P\in\Sigma_0;\ v_1=\phi_1>0,\ P\in\Sigma_1;$$
$$v_0=v_1=0, \frac{\partial v_0}{\partial N}=\frac{\partial v_1}{\partial N},\ P\in\Sigma.$$

The solution of this problem is reduced to the solution of the Dirichlet problem. *C. G. Maple* (Ames, Ia.).

Montaldo, Oscar. Sul primo problema di valori al contorno per l'equazione del calore. Rend. Sem. Fac. Sci. Univ. Cagliari **25** (1955), 1–14.

B. Pini [Rend. Sem. Mat. Univ. Padova **23** (1954), 422–434; MR **16**, 485] proved the existence of a "generalized" solution of the first boundary-value problem associated with the equation $L(u)\equiv u_{xx}-u_y=0$ in case one assumes the boundary curves of the region under consideration are simply continuous; he also proved that a theorem analogous to the second theorem of Harnack holds for a sequence of solutions of $L(u)=0$. These and other results obtained by Pini are extended in the present paper to the equation $u_{x_1x_1}+\cdots+u_{x_nx_n}-u_y=0$. *F. G. Dressel* (Durham, N.C.).

Gagliardo, Emilio. Problem al contorno generalizzato per l'equazione del calore. Ricerche Mat. **4** (1955), 74–94.

Let D denote the set of points (x, y) defined by the inequalities

$$c\leqq y\leqq d,\quad \alpha(y)\leqq x\leqq\beta(y)\quad (\alpha(y)<\beta(y)),$$

where the functions α, β belong to class C'. Use γ to denote that part of the boundary of D consisting of the segment $\gamma_1\equiv(y=c,\ \alpha(c)\leqq x\leqq\beta(c))$ and the curves $\gamma_{21}\equiv(x=\alpha(y),\ c\leqq y\leqq d)$; $\gamma_{22}\equiv(x=\beta(y),\ c\leqq y\leqq d)$. If $g=g(x, y)$ is absolutely continuous in each variable separately, g_x is absolutely continuous as a function of x, and further g_x, g_y, g_{xx} belong to L^2 in D, then g will be said to belong to class A. Consider the following boundary-value problem for the heat equation

$$(1)\qquad u_{xx}-u_y=f(x, y)\quad \text{(almost everywhere in } D),$$

$$(2)\qquad \lim_{x\to\alpha(y)} u(x, y)=w_{21}(y)\quad \lim_{x\to\beta(y)} u(x, y)=w_{22}(y),$$
$$\lim_{y\to c} u(x, y)=w_1(x),\quad \text{(almost everywhere on } \gamma).$$

The author proves that in the class A there exists a unique function $u=u(x, y)$ satisfying conditions (1) and (2) if $f(x, y)$ belongs to L^2 in D and if the assigned boundary values $w_{21}(y)$, $w_{22}(y)$, $w_1(x)$ are absolutely continuous functions whose first derivatives belong to L^2. It is also assumed that $w_1(\alpha(c))=w_{21}(c)$ and $w_1(\beta(c))=w_{22}(c)$.
F. G. Dressel (Durham, N.C.).

Budak, B. M. On the solution of boundary problems of parabolic type. Vestnik Moskov. Univ. **10** (1955), no. 8, 33–38. (Russian)

Theorem: For $i=1, 2, 3$, let $U_i(t, x)$ be a solution of the problem: (1) $[\partial/\partial t+f_i(t, x, \partial/\partial x)]U_i=0$ in D_i for $0<t<\infty$; (2) $U_i=\Phi_i$ in D_i for $t=0$; (3) $[\Lambda_i(t, x, \partial/\partial x)]U_i=0$ on Σ_i for $0<t<\infty$; where D_i is a domain in Euclidean n_i-space with a "piecewise smooth" (this term is defined and discussed in detail in the paper) boundary Σ_i, and f_i and

Λ_i are linear in x, t, and derivatives of arbitrary order in $x_1, \cdots, x_{n_i}$. The the product $U=U_1U_2U_3$ defined on $D=D_1\times D_2\times D_3$ is a solution of the problem: (1) $[\partial/\partial t+f_1+f_2+f_3]U=0$ in D for $0<t<\infty$; (2) $U=\Phi_1\Phi_2\Phi_3$ in D for $t=0$; (3) $\Lambda_1(U)=0$ on $\Gamma_1=\Sigma_1\times D_2\times D_3$ for $0<t<\infty$, and analogously for $i=2, 3$.

The theorem can be used to solve certain problems for the heat equation. *J. Cronin* (Southbridge, Mass.).

Szarski, J. Sur la limitation et l'unicité des solutions d'un système non-linéaire d'équations paraboliques aux dérivées partielles du second ordre. Ann. Polon. Math. 2 (1955), 237–249 (1956).

The non-negative continuous function $\sigma(t, y)$ on the set $0<t<T$, $y\geqq 0$, is assumed to be such that the only solution of $dy/dt=\sigma(t, y)$, $\lim_{t\to t_1} y(t)=0$, $0\leqq t_1<t$, is $y\equiv 0$. For short, set

$$z_j{}^i=\partial z^i/\partial x^j,\quad z_{jk}{}^i=\partial^2 z^i/\partial x^j\partial x^k,$$

$$f_i(t; x_1, \cdots, x_n; z^1, \cdots, z^m; z_1{}^i, \cdots, z_n{}^i; z_{11}{}^i, \cdots, z_{nn}{}^i)\equiv f_i(t; x; z; z_j{}^i; z_{jk}{}^i)\quad (i=1, \cdots, m;\ j, k=1, \cdots, n).$$

If for all values of r_{jk}, $\bar r_{jk}$ such that $\sum_{j,k=1}^n (rl_k-\bar r_{jk})\lambda_j\lambda_k\leqq 0$ (all $\lambda_1, \cdots, \lambda_n$) each f_i satisfies the inequality

$$f_i(t; x; z; p_j{}^i; r_{jk})-f_i(t; x; z; p_j{}^i; \bar r_{jk})\leqq 0,$$

then the set of equations

$$(*)\qquad \frac{\partial z^i}{\partial t}=f_i(t; x; z; x_j{}^i; z_{jk}{}^i)$$

is referred to as a parabolic system. (Note that f_1 does not depend on the derivatives of $z^1, \cdots, z^{i-1}, z^{i+1}, \cdots, z^m$.) Let I denote the interval $0<t<T$, G an open and bounded domain in the space of the variables $x_1, \cdots, x_n$, and $R=I\times G$ the topological product of I and G. Finally let R^* denote that part of the boundary of R which lies below $t=T$. For several boundary-value problems associated with (*) the author gives conditions on the f_i that will insure the existence of at most one solution. We note here only one example of his results. If the f_i satisfy the inequalities

$$|f_i(t; x; u; p_j{}^i; q_{jk}{}^i)-f_i(t; x; v; p_j{}^i; q_{jk}{}^i)|\leqq\sigma(t, \max_j|u_j-v_j|),$$

then, in the class of functions which belong to C'' on R and C^0 on the closure of R, there is at most one solution in R of the parabolic system (*) taking on preassigned values on R^*. [This is a generalization of a theorem of L. Giuliano, Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) 12 (1952), 260–265; MR 13, 948.]

F. G. Dressel (Durham, N.C.).

★**Fichera, G. Alcuni recenti sviluppi della teoria dei problemi al contorno per le equazioni alle derivate parziali lineari.** Convegno Internazionale sulle Equazioni Lineari alle Derivate Parziali, Trieste, 1954, pp. 174–227. Edizioni Cremonese, Roma, 1955. 3000 Lire.

Two approaches to the solution of linear partial differential equations, elliptic and parabolic, are developed and applied to a variety of problems. The first method is based on Theorem I: Let M_1, M_2 be linear homomorphisms from an abstract linear space V over the real (complex) numbers into the Banach spaces $\mathcal{B}_1$ and $\mathcal{B}_2$, respectively, which are over the real (complex) numbers. Let Φ be a linear functional defined on $\mathcal{B}_1$. Then a necessary sufficient condition that there exist a linear functional Ψ defined on $\mathcal{B}_2$ such that $\Phi[M_1(v)]=\Psi[M_2(v)]$ for all $v\in V$ is that there exist a constant K such that for all $v\in V$, $\|M_1(v)\|\leqq K\|M_2(v)\|$.

Theorem I is applied to solving the Dirichlet problem for the equation

$$(*)\qquad \mathcal{E}(u)\equiv\sum_{h,k}^{1,n}\frac{\partial}{\partial x_k}\left(a_{hk}\frac{\partial u}{\partial x_h}\right)+\sum_{h=1}^n b_h\frac{\partial u}{\partial x_h}+cu=f,$$

($a_{hk}=a_{kh}$, and $\sum_{h,k}^{1,n} a_{hk}(x)\lambda_h\lambda_k$ is positive definite) in a domain A. Appropriate continuity conditions are imposed on the coefficients in $\mathcal{E}(u)$, the given boundary function, and the boundary, FA, of A. Application of Theorem I is made possible by using the maximum-minimum principle for the solutions of elliptic equations; and the existence of a fundamental solution (i.e., a function $s(x, y)$ which as a function of x is a fundamental solution of $\mathcal{E}(u)=0$ and as a function of y is a fundamental solution of the adjoint equation $\mathcal{E}^*(u)=0$) is used. Secondly, a solution of the Dirichlet problem is obtained without using the existence of the fundamental solution. Instead, the existence of a function $w(x)$ defined in $A+FA$ and with special properties is assumed. The function $w(x)$ is used to derive an integral inequality which, in turn, permits application of Theorem I. This second result is an extension of results of Lax [Proc. Amer. Math. Soc. 3 (1952), 526–531; MR 14, 470] and Garabedian and Shiffman [Trans. Amer. Math. Soc. 76 (1954), 288–299; MR 15, 711]. The Neumann problem for (*), a problem with mixed boundary conditions for the equation $\mathcal{E}(u)=0$, and a boundary-value problem for a linear parabolic equation are also solved by applying Theorem I.

The second method is a Hilbert-space method. A domain A with a certain type of well-behaved boundary FA is considered, and the Dirichlet problem solved for a homogeneous elliptic equation. The Hilbert-space inner product based on the Dirichlet integral is introduced in the usual way. Let $\bar C^{(1,2)}(A)$ be the set of functions u defined in A, satisfying certain continuity and summability conditions, and the limiting values of which at FA (in a special sense too complicated to be described here) agree with the given boundary function in the Dirichlet problem. It is proved that if $\bar C^{(1,2)}(A)$ is non-empty, the Dirichlet problem has a unique solution and that this solution is the function in $\bar C^{(1,2)}(A)$ with minimum norm. In order to use this approach, some theorems of independent interest about the behavior near FA of functions defined in A and satisfying certain continuity and summability conditions are proved.

The author states that the problems considered are intended to illustrate the methods rather than to indicate the full scope of their applications, and a monograph giving more extensive developments is promised.

J. Cronin (Southbridge, Mass.).

Heinz, Erhard. Über die Eindeutigkeit beim Cauchyschen Anfangswertproblem einer elliptischen Differentialgleichung zweiter Ordnung. Nachr. Akad. Wiss. Göttingen. IIa. **1955**, 1–12.

It has been an open problem since the work of Carleman [C. R. Acad. Sci. Paris **197** (1933), 471–474] in two dimensions to prove the unique continuation property for solutions of elliptic partial differential equations. This asserts that two solutions of such an equation which agree on an open set, or have the same Cauchy data on an arc, are identical; for analytic equations this follows from the fact that solutions are analytic. Extending the work of C. Müller [Comm. Pure Appl. Math. **7** (1954),

505–515; MR 16, 42], the author establishes the unique continuation property for equations with Laplacian Δ as leading part. He proves in fact the Theorem: Let u be a twice continuously differentiable function in the unit sphere in n space satisfying: (i) for some constant K and every concentric sphere S_r of radius $r\leqq 1$, u satisfies

$$(*)\qquad \int|\Delta u|^2ds\leqq K\int(|\text{grad } u|^2+r^{-2}|u|^2))ds,$$

where ds represents elements of area on S_r, and integration extends over the surface of S_r; (ii) both sides of (*) vanish at the origin faster than any power of r. Then $u\equiv 0$ in some neighborhood of the origin.

The theorem is proved by means of the following inequality: there exists a constant C such that

$$\iint|x|^{-k}(|\text{grad } u|^2+|x|^{-2}|u|^2)dx\leqq C\iint|x|^{-k}|\Delta u|^2dx$$

holds for all real k and all functions u satisfying (ii) above, and vanishing together with first derivatives at $|x|=1$. Here integration is over the full unit sphere. The inequality, which is proved with the aid of spherical harmonics, is a generalization of an inequality of Rellich [Perturbation theory of eigenvalue problems, Inst. Math. Sci., New York Univ., 1953]. *L. Nirenberg.*

Kamenomostskaya, S. L. The first boundary problem for equations of elliptic type with a small parameter with the highest derivatives. Izv. Akad. Nauk SSSR. Ser. Mat. **19** (1955), 345–360. (Russian)

The equation $\varepsilon\Delta u+Au_x+Bu_y+Cu=D$ where A, B, C, and D are functions of (x, y) is considered in a region G with boundary Γ. The boundary values of u_ε are assigned on Γ by a given function ϕ. It is assumed in $G+\Gamma$ that $C<0$. The part of Γ where the characteristic curves $dx/dt=A$, $dy/dt=B$ for increasing t leave the region by crossing the boundary Γ from inside to outside is denoted by $\tilde{\Gamma}_1$. The author's theorem is that in a region of $G+\Gamma$ free of characteristics which reach Γ but do not cross it at once from interior to exterior and free of points of $\Gamma-\tilde{\Gamma}_1$, $u_\varepsilon(x, y)-U(x, y)\to 0$, where U is the solution of $AU_x+BU_y+CU=D$ which on $\tilde{\Gamma}_1$ assumes the values ϕ. It is not assumed that $t^2+B^2>0$ throughout $C+\Gamma$.

N. Levinson (Cambridge, Mass.).

Hartman, Philip; and Wintner, Aurel. On a comparison theorem for selfadjoint partial differential equations of elliptic type. Proc. Amer. Math. Soc. **6** (1955), 862–865.

Consider two elliptic differential equations

$$\sum_1^n\frac{\partial}{\partial x_\alpha}\Big(\sum_1^n G_j^{\alpha\beta}\frac{\partial u}{\partial x_\beta}\Big)+f_ju\quad(j=\text{I, II})$$

defined in a bounded domain T. The first equation is said to dominate the second if the difference matrix $(G^{\text{I}}_{\alpha\beta}-G^{\text{II}}_{\alpha\beta})$ is positive semi-definite and $f_{\text{II}}\leqq f_{\text{I}}$. Here $(G_{\alpha\beta}{}^j)$ are the inverses of the positive definite matrices $(G^{\alpha\beta}{}_j)$. If $f_{\text{II}}\equiv f_{\text{I}}$, then it is assumed that at some point at which the function f_{I} does not vanish the above difference matrix is definite. The coefficients of the matrices G_j and the functions f_j are assumed to be continuous. Theorem: If equation (I) dominates equation (II) and if (I) has a solution which vanishes on the boundary of T but does not vanish on any open subset, then every solution of the second equation must vanish somewhere in T. *L. Bers.*

Hartman, Philip; and Wintner, Aurel. On uniform Dini conditions in the theory of linear partial differential equations of elliptic type. Amer. J. Math. **77** (1955), 329–354.

The authors consider linear elliptic differential equations of the form (1) $(A_1u_x+A_2u_y)_x+(A_3u_x+A_4u_y)_y+Cu=F$. In seeking C^1 (C^2) solutions u it is customary to require that the coefficients A_i (and their first derivatives DA_i) satisfy a Hölder condition, and that C, F are continuous (Hölder continuous). The first derivatives Du (second derivatives D^2u) are then also Hölder continuous. Assuming merely that the coefficients A_i are uniformly Dini continuous, i.e. that $|A_i(P)-A_i(Q)|\leqq\text{const.}\times\alpha(|PQ|)$, for all points P, Q, where $\alpha(r)$ is a monotonic increasing function satisfying Dini's condition $\int_{0+}\alpha(r)r^{-1}dr<\infty$, the authors construct C^1 solutions in a small circle, with given smooth boundary values. The solutions are obtained, via a compactness argument, from an a priori estimate for the modulus of continuity of the first derivatives Du of a solution u:

(2) $|Du(P)-Du(Q)|\leqq\text{const.}\times\beta(|PQ|)$, where

$$\beta(r)=(r+\alpha(r))\log(R/r)+\int_0^{4r}\alpha(\varrho)\varrho^{-1}d\varrho+r\int_r^R\alpha(\varrho)\varrho^{-2}d\varrho.$$

(If in addition C, F and DA_i are uniformly Dini continuous the solutions are in C^2.) It follows that on a Riemann surface with given Dini continuous metric tensor g_{ij} one may find C^1 isothermal local coordinates.

The estimate (2) is derived along the lines of E. Hopf's treatment of equations with Hölder continuous coefficients [Math. Z. **34** (1931), 194–233], with the aid of Green's function for Laplace's equation. It is interesting that a solution of (1), with A_i Dini continuous, need not have Dini continuous first derivatives; this is shown by an example. The discussion extends also to equations in higher dimensions, and, as far as interior differentiability and estimates are concerned, to equations of higher order.

L. Nirenberg (New York, N.Y.).

Principalli, Maria Luisa. Sul sistema di equazioni lineari alle derivate parziali, relativo all' equilibrio delle volte cilindriche. Ann. Scuola Norm. Sup. Pisa (3) **8** (1954), 157–291 (1955).

In this comprehensive paper, the author summarizes the work of G. Fichera and other writers in handling the Dirichlet problem for second-order linear elliptic equations and then proceeds to extend these methods to get many results for systems of 2 elliptic equations, for the biharmonic equation, and finally for a system important in the theory of elasticity:

$$\Delta_2u_1+k\frac{\partial}{\partial x}\Big(\frac{\partial u_1}{\partial x}+\frac{\partial u_2}{\partial y}\Big)+\frac{\partial(bu_3)}{\partial x}=F_1;$$

$$\Delta_2u_2+k\frac{\partial}{\partial y}\Big(\frac{\partial u_1}{\partial x}+\frac{\partial u_2}{\partial y}\Big)+\frac{\partial(cu_3)}{\partial y}=F_2;$$

$$bu_{1,x}+cu_{2,x}+\Delta_4u_3+au_3=F_3,$$

where k is a real constant and a, b, c, F_1, F_2, F_3 are functions of (x, y) defined in a bounded region A. She establishes existence and uniqueness theorems, results concerning the Green's functions, and in many cases derives the fundamental inequalities necessary to show that certain transformations are totally continuous. Classes of functions and regions to be considered are carefully defined and hypotheses in all theorems are well-stated. A more general system of equations than that

given above is considered to some extent in the last chapter. The only boundary conditions considered are that the solution assume given values, on the boundary of A, or in the case of the above system, that $u_1=u_2=u_3=\partial u_3/\partial v=0$ on the boundary of A. It is to be hoped that in later papers some other boundary problems will be studied.
D. L. Bernstein (Rochester, N.Y.).

Albrecht, Julius. Eine einheitliche Herleitung der Gleichungen von Trefftz und Galerkin. Z. Angew. Math. Mech. **35** (1955), 193–195.

Consider an N-dimensional region B with boundary Γ. Let $L[\ ]$ denote any linear self-adjoint partial differential operator over B, second-order, totally elliptic. The author seeks an approximate solution of $L[u]=r$ over B; boundary conditions of either the first, second, or third ("mixed") kind are specified on Γ: $\Lambda[u]=s$. B, Γ, the operators L, Λ and the functions r, s (as well as the w's considered below) are restricted so that Green's formula holds. The approximate solution has the form

$$w=w_0+\sum_{\mu=1}^{M} c_\mu w_\mu.$$

Here w is either a Trefftz trial function ($L[w_0]=r$, $L[w_\mu]=0$) or a Galerkin trial function ($\Lambda[w_0]=s$, $\Lambda[w_\mu]=0$). The author now introduces a metric based on the positive-definite Green's matrix for the operator L. Choosing the constants c_μ so that $\|w-u\|$ is stationary in this metric, he derives the "Trefftz approximation" and the "Galerkin approximation", generalized to N dimensions. The paper concludes with several examples. For related work, see J. B. Diaz, Collect. Math. **4** (1951), 3–49 [MR **14**, 1084] and references given there.
M. A. Hyman (Philadelphia, Pa.).

Pólya, Georges. Sur les fréquences propres des membranes vibrantes. C. R. Acad. Sci. Paris **242** (1956), 708–709.

The membrane problem $\Delta u+\lambda u=0$ is considered on a domain D. The eigenvalues of the fixed membrane (boundary condition $u=0$) arranged in non-decreasing order are called $\lambda_1, \lambda_2, \cdots$. Those of the free membrane ($\partial u/\partial n=0$) are called $\mu_1, \mu_2, \cdots$. It is known [Weyl, Math. Ann. **71** (1912), 441–479; Courant and Hilbert, Methoden der mathematischen Physik, Bd. 1, 2nd ed., Springer, Berlin, 1931, pp. 353–387] that the ratios λ_k/k and μ_{k+1}/k tend to $4\pi/A$ as $k\to\infty$, where A is the area of D. The author shows that if D is a domain such that the whole plane may be covered by non-overlapping copies of D, then

$$\lambda_k/k\geqq 4\pi/A \quad (k=1, 2, \cdots).$$

Under a stronger restriction on D it is proved that

$$\mu_{k+1}/k\leqq 4\pi/A \quad (k=1, 2, \cdots).$$

The author has conjectured that these inequalities hold for all domains D [Polya, Patterns of plausible inference···, v. 2, Princeton, 1954, pp. 51–53; MR **16**, 556].
H. F. Weinberger (College Park, Md.).

Hong, Imsik. On the null-set of a solution for the equation $\Delta u+k^2u=0$. Kōdai Math. Sem. Rep. **7** (1955), 53–54.

Let D be a domain in the plane and M a subset of logarithmic capacity zero. The author shows that if u is a bounded function which satisfies $\Delta u+k^2u=0$ on $D-M$, then u satisfies this equation in D. The author also states a second proposition which is somewhat unclear but seems to be a statement of the well-known fact that a solution of this equation must be both positive and negative in every neighborhood of a zero.
H. L. Royden.

★**Pleijel, Åke. On Green's functions and the eigenvalue-distribution of the three-dimensional membrane equation.** Tolfte Skandinaviska Matematikerkongressen, Lund, 1953, pp. 222–240 (1954). 25 Swedish crowns (may be ordered from Lunds Universitets Matematiska Institution).

The author extends to the three-dimensional case an earlier paper on the asymptotic distribution of eigenvalues and eigenfunctions of the equation (1) $\Delta u+\lambda u=0$ in two dimensions [Ark. Mat. **2** (1954), 533–569; MR **15**, 798, 1140]. For points x, y in a bounded domain V, in E^3, having infinitely differentiable boundary S, let $-\gamma(x,y,\varkappa)$ be the regular part of the Green's function (for the Dirichlet or Neumann problem), having singularity at y, for the equation $\Delta u-\varkappa^2u=0$ in V. By a detailed analysis of the expressions obtained from the Fredholm integral equation construction of γ (this analysis relies on the smoothness of S) the author can approximate $\gamma(x, x, \varkappa)$ as $\varkappa\to\infty$, by certain expressions to within an error $O(\varkappa^{-N}e^{-2A\varkappa\eta})$; here η is the distance from x to S, N is an arbitrary large number, and $A<1$ is a constant, arbitrarily close to unity.

From this approximation follows the expansion

$$(2)\quad \int_V[\gamma(x, x, \varkappa)-\gamma(x, x, \varkappa_0)]dx=C\log\varkappa+\sum_0^{m-1} a_\nu\varkappa^{-\nu}+O(\varkappa^{-m})$$

for m arbitrarily large. The constants C, a_ν are different in the Dirichlet and Neumann problems, C being respectively $-S/8\pi$ and $S/8\pi$. (2) is applied to study the asymptotic behavior of the eigenvalues λ_n of (1). It is shown that $\varkappa^2\sum^\infty \lambda_n^{-1}(\lambda_n+\varkappa^2)^{-1}-V\varkappa/4\pi$ has an expansion similar to (2); here V is the volume. In addition, using methods of the paper cited above the author extends results of Carleman [Åttonde Skand. Matematiker kongressen, Stockholm, 1934, Ohlsson, Lund, 1935, pp. 34–44] by deriving the following analytic continuation of the Dirichlet series

$$\sum_1^\infty \lambda_n^{-z}=\frac{V}{4\pi^2(z-\frac{3}{2})}+\frac{C}{2(z-1)}+\sum_1^{m-1}\frac{a_{2\nu-1}}{\pi(2+\frac{1}{2}(2\nu-3))}+\chi_m(z)$$

where $\chi_m(z)$ is analytic for $\operatorname{Re} z>(3/2-m)$.
L. Nirenberg (New York, N.Y.).

★**Pleijel, Å. On the problem of improving Weyl's law for the asymptotic eigenvalue distribution.** Convegno Internazionale sulle Equazioni Lineari alle Derivate Parziali, Trieste, 1954, pp. 69–75. Edizioni Cremonese, Roma, 1955. 3000 Lire.

Exposé des résultats de l'auteur quant à l'amélioration de la formule de H. Weyl donnant le comportement asymptotique du nombre des valeurs propres inférieures ou égales à t pour le problème de vibration posé pour l'équation $(\Delta+\lambda)u=0$, dans un ouvert borné de R^3 à frontière assez régulière, avec les conditions aux limites de Dirichlet [pour le développement des calculs, voyez l'oeuvre analysée ci-dessus]. La formule de H. Weyl peut s'obtenir par un procédé dû à T. Carleman [Åttonde Skand. Matematikerkongressen, Stockholm, 1934, Ohlsson, Lund, 1935, pp. 34–44]. L'auteur adopte le même point de départ que Carleman mais complète son résultat en utilisant une évaluation plus poussée du comportement, lorsque k est grand, du noyau compensateur de

$\Delta-k$ pour le problème de Dirichlet, évaluation qui joue un rôle important dans les considérations de Carleman.

Le travail contient une critique des résultats obtenus par R. H. Bolt et G. M. Roe [Roe, J. Acoust. Soc. Amer. 13 (1941), 1–7] donnant une estimation du reste de la formule de H. Weyl pour un parallélépipède rectangle.

H. G. Garnir (Liège).

Martin, A. I.; and Titchmarsh, E. C. On the Parseval formula in the theory of eigenfunction expansions arising from differential equations. Quart. J. Math. Oxford Ser. (2) 6 (1955), 197–206.

In the n-dimensional real space E_n consider the equation $-\Delta u+q(x)u=\lambda u$, where Δ is the Laplacian and q is a real function in E_n. Let $G=G(x, \xi, \lambda)$ be any Green's function associated with this equation in E_n; it is a limit of Green's functions for appropriate problems on compact subsets of E_n, and hence need not be unique. Let $H(x, \xi, \mu)=\lim \int^\mu \Im G(x, \xi, v+i\varepsilon)dv$ $(\varepsilon\to 0)$. It is shown that if $f, g \in L^2(E_n)$ and are real then

$$\int fg=\lim_{\mu\to\infty}\frac{1}{\pi}\iint[H(x, \xi, \mu)-H(x, \xi, -\mu)]f(\xi)g(x)dxd\xi,$$

where the integrations are over E_n. This is the Parseval formula of the title. Now suppose that f satisfies the conditions for the expansion theorem as proved by Titchmarsh [Proc. London Math. Soc. (3) 1 (1951), 1–27; MR 13, 241], and assume $f_x, f_y \in L^2(E^2)$, $qf^2 \in L^1(E_2)$. Then it is shown that

$$D(f)\equiv\int_{E_2}(f_x^2+f_y^2+qf^2)=\int_{-\infty}^{\infty}\lambda dJ(f, \lambda),$$

where

$$J(f, \lambda)=\frac{1}{\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}H(x, \xi, \lambda)f(x)f(\xi)dxd\xi.$$

Finally it is shown that in certain cases the inequality $\int_{-\infty}^{\infty}\lambda dJ(f, \lambda)\leqq D(f)$ can be proved under wider conditions than those assumed for the corresponding equality.

E. A. Coddington (Copenhagen).

Zaidman, S.; et Boboc, N. L'unicité du problème de Dirichlet pour des équations de type elliptique. Acad. R. P. Romîne. Bul. Şti. Secţ. Şti. Mat. Fiz. 6 (1954), 839–846. (Romanian. Russian and French summaries)

The authors claim to prove the following statement: $u\equiv 0$ is the only solution with zero boundary values of the equation $\Delta u+A(x, y)u=0$ in a simply connected domain $\mathfrak{D}$, provided that the boundary may be expressed, in polar coordinates, by an equation $r=r(\theta)$, with $r(\theta)$ continuously differentiable, and provided that (Area $\mathfrak{D}$)$\cdot\iint A^2dxdy\leqq 1$. This statement is an analogue of a one-dimensional result due to G. Borg [Amer. J. Math. 71 (1949), 67–70; MR 10, 456]. The proof however appears to be incorrect (the last inequality on p. 842 is false).

L. Nirenberg (New York, N.Y.).

Levitan, B. M. On expansion in eigenfunctions of the Schrödinger operator in the case of a potential increasing without bound. Dokl. Akad. Nauk SSSR (N.S.) 103 (1955), 191–194; erratum 105 (1955), 620. (Russian)

The equation $\Delta u+\{\lambda-q(x_1, x_2, x_3)\}u=0$ is considered in three-dimensional Euclidean space E_3. Rather complicated conditions are imposed on q which make it behave roughly as a polynomial at infinity. Under these conditions, if $\int_{E_3}[f^2(x)/q^\tau(x)]dx<\infty$ $(\tau\geqq 0)$, it is shown that

$$\lim_{\lambda\to\infty}\sum_{\lambda_n<\lambda}\left(1-\frac{\lambda_n}{\lambda}\right)^s c_n\omega_n(x)=f(x)$$

at each continuity point of f. Here $s>\beta+1$, where β is a constant involved in the restrictions on q, the ω_n are the eigenfunctions, and the c_n are the Fourier coefficients of f with respect to the ω_n. A brief outline of the proof is given. It involves an asymptotic formula which is an extension of one obtained by Titchmarsh for E_2 [Proc. London Math. Soc. (3) 3 (1953), 153–169; MR 15, 229]. In the erratum Levitan states that he was incorrect in his assertion that there is a mistake in a derivation of a particular case of this formula given by D. Ray [Trans. Amer. Math. Soc. 77 (1954), 299–321; MR 16, 593].

E. A. Coddington (Copenhagen).

Tanimura, Masayoshi. On the solution of some mixed boundary problems. I. Different boundary conditions to each consecutive semi-infinite surface. Tech. Rep. Osaka Univ. 5 (1955), 77–102.

Linear boundary-value problems in partial differential equations in unbounded regions, where a Dirichlet condition is prescribed on one half of a plane boundary of infinite extent while a Neumann condition is prescribed on the other half of that boundary, are considered. Examples are also given in which the Neumann condition is replaced by a condition of the third type, and in which the plane boundary is replaced by a cylindrical one. The object is to establish integral equations in the boundary values of the unknown function and its normal derivatives, on the boundary mentioned above. Solutions of the integral equations would make it possible to replace the boundary conditions on the two half-planes by one of the Dirichlet type on the entire plane. Integral equations are set up for sample problems here by formal manipulations with Laplace transformations. No distinction is made between one-sided and two-sided transformations, and a number of steps are left unexplained and unjustified.

R. V. Churchill (Ann Arbor, Mich.).

Mel'nik, S. I. Some estimates for a biharmonic function. Dokl. Akad. Nauk SSSR (N.S.) 104 (1955), 352–355. (Russian)

Let B be a bounded, simply-connected region in the (x, y)-plane. Let the boundary Γ_B of B possess continuous curvature $\varrho(s)^{-1}$. Let l denote the total arc-length of Γ_B. We place the origin $(0, 0)$ inside B, and require that $dR/dn|_{\Gamma_B}=\cos(R, n)\neq 0$, where $R^2=x^2+y^2$. Let $f_1(s)$, $f_2(s)$, and $f_1'(s)$ belong to L^2, with $\int_0^l f_1^2(s)\, ds=M$. Define the biharmonic function W by the boundary conditions

$$W=f_1(s) \text{ on } \Gamma_B,\quad \frac{dW}{dn}=f_2(s) \text{ on } \Gamma_B.$$

Estimates are now given for W and for its derivatives up to order two, inclusive. We reproduce the estimates on the second derivatives:

$$\left|\frac{\partial^2W}{\partial x^2}\right|, \left|\frac{\partial^2W}{\partial y^2}\right|\leqq\frac{k_1}{\pi(1-k)}\Big\{2\Big[\int_0^l\frac{ds}{r^2_{p,Q}}\Big]^{\frac12}+ 12\max R\Big[\int_0^l\frac{ds}{r^4_{p,Q}}\Big]^{\frac12}+9\max R^2\Big[\int_0^l\frac{ds}{r^6_{p,Q}}\Big]^{\frac12}\Big\}+ 9\,\frac{M+k_2}{\pi(1-k)}\Big[\int_0^l\frac{ds}{r^6_{p,Q}}\Big]^{\frac12},$$

$$\left|\frac{\partial^2 W}{\partial x \partial y}\right| < \frac{k_1}{\pi(1-k)}\left\{4\left[\int_0^l \frac{ds}{r^4_{p,Q}}\right]^{\frac{1}{2}} + 13 \max R\left[\int_0^l \frac{ds}{r^6_{p,Q}}\right]^{\frac{1}{2}}\right\} + 13\,\frac{M+k_2}{\pi(1-k)}\left[\int_0^l \frac{ds}{r^6_{p,Q}}\right]^{\frac{1}{2}}.$$

In these formulas, $W=W(Q)$, Q is an interior point of B, p is a point on the boundary, $ds=ds_p$, and k, k_1, k_2 and are defined as follows. Let $K(s, s_0)=\pi^{-1}d \ln r_{pq}^{-1}/dn_p$; then

$$k=\min_{(\phi,\phi=1)} \int_0^l\int_0^l K(s, s_0)\phi(s)\phi(s_0)dsds_0.$$

Let v be the harmonic function in B assuming the boundary value $v=f_1(s)$ on Γ_B. Define

$$\theta(s)=f_2(s)-\left.\frac{dv}{dn}\right|_{\Gamma_B}.$$

Then

$$k_1^2=\frac{1}{\min(dR^2/dn)^2}\int_0^l \theta^2(s)ds,$$

and

$$k_2^2=\frac{\max R^4}{\min(dR^2/dn)^2}\int_0^l \theta^2(s)ds.$$

The proof is achieved by conformal mapping onto the unit circle, representation of the biharmonic function in terms of harmonic functions, use of Poisson's integral, and estimating the L^2 norm of solutions of Fredholm equations. *R. B. Davis* (Durham, N.H.).

Agaev, G. N. On the analytic character of the solutions of a system of partial differential equations. Akad. Nauk Azerbaĭdžan. SSR. Trudy Inst. Fiz. Mat. **3** (1948), 73–84. (Russian. Azerbaijani summary)

The author considers the system of nonlinear elliptic differential equations

$$\Delta^{n_i}u_i=f_i(x, y; u_1, u_2, \cdots, u_{1,xx}, \cdots)\quad (i=1, 2, \cdots, m),$$

where the right-hand sides depend upon the derivatives of the ith unknown function up to the order $2n_i-1$. The author claims to have proved that all sufficiently regular solutions of this system are real analytic. The reviewer is unable to verify the proof since it seems that on page 81 the author asserts the analyticity of a uniform limit of real analytic functions. *L. Bers* (New York, N.Y.).

Agaev, G. N. Solution of a nonlinear boundary problem for a system of polyharmonic equations. Akad. Nauk Azerbaĭdžan. SSR. Trudy Inst. Fiz. Mat. **4-5** (1952), 24–30. (Russian. Azerbaijani summary)

This paper deals with the homogeneous boundary-value problem for the system of equations

$$(1)\quad \Delta^{m_i}u_i=f_i(x_1, x_2, \cdots, x_n, u_1, u_2, \cdots, u_N)\quad (i=1, 2, \cdots, N)$$

defined in a domain D. The solution is to satisfy the boundary conditions

$$(2)\quad u_i|_s=0,\ \Delta u_i|_s=0, \cdots, \Delta^{m_i-1}u_i|_s=0\quad (i=1, 2, \cdots, N)$$

on the boundary S. Using the Green's function of the Laplace equation this boundary-value problem is reduced to a nonlinear integral equation. The latter is considered in an appropriate Banach space and is solved by the principle of contracting mappings (or, which is the same, by successive approximations). In this way it is shown that the boundary-value problem has a solution if the domain has sufficiently small measure, the right-hand sides satisfy a Lipschitz condition with respect to the unknown functions and are sufficiently small. *L. Bers* (New York, N.Y.).

Gabib-zade, A. Š. On a general complex representation of single-valued real solutions of a system of polyharmonic equations. Akad. Nauk Azerbaĭdžan. SSR. Trudy Inst. Fiz. Mat. **4-5** (1952), 38–56. (Russian. Azerbaijani summary)

The aim of this paper is to obtain a general representation for singlevalued solutions of the system of differential equations

$$(S)\quad L_{n_k}(u)\equiv\Delta^{n_k}u_k+\sum_{j=1}^{m}\sum_{l=1}^{\nu_{kj}}\sum_{0\le p+q\le l} A_{jlpq}{}^k(x, y)\frac{\partial^{p+q}}{\partial x^p\partial y^q}(\Delta^{\nu_{kj}-l}u_j)=0,\quad (k=1, 2, \cdots, m)$$

in a multiply connected domain T. The functions $A(x, y)$ are assumed to be entire analytic. Using the methods of Vekua, Halilov [C. R. (Dokl.) Acad. Sci. URSS (N.S.) **51** (1946), 171–173; MR **8**, 68] gave a general complex representation of solutions of (S) in terms of analytic functions. The author uses this representation in order to obtain a representation which will always give single-valued solutions. The final formulas are too complicated to be reproduced here. *L. Bers* (New York, N.Y.).

Galustyan, S. B. On solution of a mixed problem for the equation of vibration of a bar. Akad. Nauk Azerbaĭdžan. SSR. Trudy Inst. Fiz. Mat. **4-5** (1952), 57–89. (Russian. Azerbaijani summary)

Let $u(x, t)$ be a function satisfying the equation $u_{xxxx}+u_{tt}=0$ in a closed region of the xt-plane bounded by the two segments $t=h$ and $t=l$ (h and l constant) and the two curves $x=\gamma_j(t)$ $(j=1, 2)$. It is required to find such a u which satisfies the boundary conditions $u=f_j(t)$, $u_x=\phi_j(t)$ on $\gamma_j(t)$; $f_j(h)=\phi_j(h)=0$; $u(x, l)=u_t(x, l)=0$. Appropriate existence and continuity conditions on the functions and their derivatives are assumed. It is shown that if u has the form $u=u_1+u_2$, where u_1, u_2 satisfy respectively the equations $u_t+iu_{xx}=0$, $u_t-iu_{xx}=0$, then the problem is equivalent to the solution of a system of four integral equations of Volterra type, in which two may alternatively be replaced by integro-differential equations. The system is solved for the case where $\gamma_j(x)=x_j$ (constant) and $h=0$, and from this $u(x, t)$ is obtained for the case $x_1=0$, $x_2=\infty$, $f_2(t)=\phi_2(t)=0$ (transverse vibration of an infinite bar). When the curve $\gamma_2(t)$ recedes to infinity and the boundary conditions are imposed only on $\gamma_1(t)$, the number of equations in the system is diminished by two. *R. N. Goss* (San Diego, Calif.).

Gal'pern, S. A. Cauchy's problem for an equation of S. L. Sobolev's type. Dokl. Akad. Nauk SSSR (N.S.) **104** (1955), 815–818. (Russian)

This paper considers explicit representations for solutions of

$$(\partial/\partial t)^2\Delta u=-(\partial/\partial x_n)^2u+f(t, x_1, x_2, \cdots, x_n)$$

with initial conditions

$$u=\phi \text{ when } t=0,\ (\partial u/\partial t)=\psi \text{ when } t=0.$$

Here Δu denotes the n-dimensional Laplacian of u,

$$\Delta u\equiv[(\partial/\partial x_1)^2+(\partial/\partial x_2)^2+\cdots+(\partial/\partial x_n)^2]u.$$

It is proved that solutions of this initial-value problem are unique, and that every sufficiently smooth solution can be represented in the form

$$u(t, x_1, \cdots, x_n) = \int \Delta^s \phi(\xi_1, \cdots, \xi_n) H^{(s)}(t, x_1-\xi_1, \cdots, x_n-\xi_n) d\xi_1 \cdots d\xi_n,$$

provided one first reduces to an equivalent problem where $\psi=0$. Here s is an integer which must be chosen in a suitable way, and an explicit expression is given for $H^{(s)}$, obtained by inverting the order of integration in a Fourier transform. Both the uniqueness and the representability follow from Green's formula, provided one has a suitable fundamental solution. Such a fundamental solution is obtained by means of the Fourier integral. This generalizes a result of Sobolev [Izv. Akad. Nauk SSSR. Ser. Mat. 18 (1954), 3–50; MR 16, 1029] who dealt with the case $n=3$. *R. B. Davis* (Durham, N.H.).

Maslennikova, V. N. Construction of a solution of Cauchy's problem for a system of partial differential equations. Dokl. Akad. Nauk SSSR (N.S.) 102 (1955), 685–688. (Russian)

Maslennikova, V. N. On mixed problems for a system of equations of mathematical physics. Dokl. Akad. Nauk SSSR (N.S.) 102 (1955), 885–888. (Russian)

The author considers the system of equations

$$(1)\quad \frac{\partial v_x}{\partial t}=v_y-\frac{\partial p}{\partial x}+F_x,\quad \frac{\partial v_y}{\partial t}=-v_x-\frac{\partial p}{\partial y}+F_y$$

$$\frac{\partial v_z}{\partial t}=-\frac{\partial p}{\partial z}+F_z,\quad \alpha^2\frac{\partial p}{\partial t}=-\frac{\partial v_x}{\partial x}-\frac{\partial v_y}{\partial y}-\frac{\partial v_z}{\partial z}+\psi$$

and the related equation, obtained from (1) by elimination of v_x, v_y, v_z:

$$(2)\quad \frac{\partial^2 \Delta p}{\partial t^2}+\frac{\partial^2 p}{\partial z^2}-\alpha^2\frac{\partial^2 p}{\partial t^2}-\alpha^2\frac{\partial^4 p}{\partial t^4}=f.$$

Here $\Delta=\partial^2/\partial x^2+\partial^2/\partial y^2+\partial^2/\partial z^2$ and v_x, v_y, v_z are components of a vector $\bar{V}$. The case $\alpha=0$ was considered by S. L. Sobolev [Izv. Akad. Nauk SSSR. Ser. Mat. 18 (1954), 3–50; MR 16, 1029] who solved the Cauchy problem for (1) and (2) explicitly, with initial data prescribed throughout xyz-space.

In the first note the solutions of Sobolev are extended to explicit solutions valid for any $\alpha\neq 0$. That is, for prescribed data $\{\bar{V}^0(x, y, z), p^0(x, y, z)\}$, a solution system $\{\bar{V}(x, y, z, t), p(x, y, z, t)\}$ of (1) is given such that $\bar{V}(x, y, z, t_0)=\bar{V}^0$, $p(x, y, z, t_0)=p^0$. Also, for prescribed data $\{\varphi_k(x, y, z)\}$ $(k=0, 1, 2, 3)$, a solution $p(x, y, z, t)$ of (2) is given such that $\partial^k p/\partial t^k|_{t=0}=\varphi_k(x, y, z)$. The construction uses the fundamental solution of the associated homogeneous equations, given explicitly by Sobolev [loc.cit.].

In the second note initial and boundary-value problems for (1) and (2) are discussed in the case $\alpha=1$. Under the assumption

$$\int_0^\infty \iiint_\Omega f^2 e^{-2\lambda_1 t} d\Omega dt<\infty \text{ for suitable } \lambda_1,$$

the author proves the existence and uniqueness of a generalized solution of (1) in a cylinder with a region Ω of xyz-space as base and generators parallel to the t-axis, satisfying the initial conditions $\bar{V}(x, y, z, t_0)=\bar{V}^0(x, y, z)$, $p(x, y, z, t_0)=p^0(x, y, z)$, and satisfying on the lateral surface of the cylinder one of the boundary conditions, $p=0$, $v_n=0$, where v_n is the normal component of $\bar{V}$. For equation (2) the author proves the existence of a unique generalized solution $p(x, y, z, t)$ such that $\partial^k p/\partial t^k|_{t=0}=0$ $(k=1, 0, 2, 3)$, and such that on the lateral surface one of the conditions $p=0$,

$$\left[\frac{\partial p}{\partial z}\cos nz+\frac{\partial^3 p}{\partial t^2\partial n}-\frac{\partial^2 p}{\partial x\partial t}\cos ny+\frac{\partial^2 p}{\partial y\partial t}\cos ny\right]=0,$$

is satisfied. The proofs use Laplace-transform methods. *R. Finn* (Los Angeles, Calif.).

See also: Friedrichs, p. 615; Shiffman, p. 632; Slobodyanskiĭ, p. 648; Souriau, p. 657; Sheldon, p. 668; Aržanyh, p. 684; Vorovič, p. 685.

Difference Equations, Special Functional Equations

Skalkina, M. A. On preservation of asymptotic stability in the passage from differential equations to the corresponding difference equations. Dokl. Akad. Nauk SSSR (N.S.) 104 (1955), 505–508. (Russian)

The author considers the problem of determining when asymptotic stability of all solutions of the differential system, $dx/dt=f(x, t)$, carries over the corresponding difference system $x(t+h)=x(t)+f(x, t)h$, $t=0, h, 2h, \cdots$. A number of results are established under certain hypotheses concerning the nature of the solutions of the differential system. *R. Bellman* (Santa Monica, Calif.).

Yates, Barbara G. The linear difference-differential equation with linear coefficients. Trans. Amer. Math. Soc. 80 (1955), 281–298.

The equation treated is linear, of arbitrary derivative order and arbitrary number of differences in the independent variable. The coefficients are linear in the independent variable. Existence and uniqueness theorems are established, as well as asymptotic solutions for large values of the independent variable. Laplace-transform techniques are used, following the methods of E. M. Wright. The analysis is intricate, requiring a careful study of the characteristic function, an exponential polynomial. *E. Pinney* (Berkeley, Calif.).

Thron, W. J. Entire solutions of the functional equation $f(f(z))=g(z)$. Canad. J. Math. 8 (1956), 47–48.

Let $g(z)$ be entire of finite order, not a polynomial, and suppose a value ϱ exists which $g(z)$ takes on only a finite number of times. Then the equation of the title has no solution $f(z)$ that is entire. *I. M. Sheffer.*

Vaughan, H. E. Characterization of the sine and cosine. Amer. Math. Monthly 62 (1955), 707–713.

The author gives all continuous and some discontinuous solutions of the functional equation $g(x-y)=g(x)g(y)+f(x)f(y)$. This equation has been entirely solved by several authors, of whom we mention W. H. Wilson [Bull. Amer. Math. Soc. 26 (1920), 300–312], J. C. H. Gerretsen [Euclides, Groningen 16 (1939), 92–99], J. G. van der Corput [ibid. 17 (1940), 55–64, 65–75], and L. Vietoris [J. Reine Angew. Math. 186 (1944), 1–15; MR 6, 271]. The special case $f(x)=g(x-s)$ is also examined. *J. Aczél* (Debrecen).

Bellman, R. Some functional equations in the theory of dynamic programming. I. Functions of points and point transformations. Trans. Amer. Math. Soc. 80 (1955), 51–71.

Bellman, Richard. Functional equations in the theory of dynamic programming. V. Positivity and quasi-linearity. Proc. Nat. Acad. Sci. U.S.A. **41** (1955), 743–746.

In I, the author investigates the functional equation

$$f(p)=\sup_{q(p)}\,(g[p,q(p)]+h[p,q(p)]f\{T[p,q(p)]\})$$

(p, q, T vectors; f, g, h scalars) with several generalizations and applications (programming, decision processes, games of survival, calculus of variations, etc.). Existence and uniqueness theorems are proved by successive-approximation processes under different preliminary suppositions.

Paper V examines under what conditions the more general equation

$$(*)\qquad f(p)=\sup_{q(p)}\{g[p,q(p)]+L(f,p,q)\}$$

(L is q linear operator of f) has the solution $f(p)=\sup_{q(p)} v_q(p)$, where $v_q(p)$ is a solution of the linear equation

$$(**)\qquad v_q(p)=g[p,q(p)]+L(v_q,p,q).$$

It is stated, that this is the case if both equations have unique solutions and if the validity of $w(p)\geqq g[p,q(p)]+L(w,p,q)$ for all p and one $q(p)$ implies $w(p)\geqq v_q(p)$. Applications are given to programming, to ordinary and partial differential equations and to the calculus of variations; of these we mention the result that the Riccati equation $dy/dt=-y^2+g(t)$, written as $dy/dt=\max_{q(t)}(q^2+2qy+g)$, has the solution

$$y=\min_{q(t)}\left\{c\exp\left(2\int_0^t g(s)ds\right)+\int_0^t \exp\left\{2\int_s^t q(u)du\right)[q(s)^2+g(s)]ds\right\}$$

(with a misprint in the text). Analogous theorems and applications are given if $\sup_{q(p)}$ is replaced by $\min_{r(p)}\max_{q(p)}$ and $q(p)$ by $q(p)$, $r(p)$ in both equations (*) and (**). No proofs are given in this paper. *J. Aczél.*

See also: Jabotinsky, p. 601; Bellman, Glicksberg and Gross, p. 641; Sheldon, p. 668.

Integral Equations, Equations in Infinitely Many Variables

Cameron, R. H.; and Shapiro, J. M. Nonlinear integral equations. Ann. of Math. (2) **62** (1955), 472–497.

The authors consider the nonlinear integral equation

$$(1)\qquad y(t)=x(t)+\int_0^t F(t,s,x(s))ds,$$

where $x(t)$ ranges over the space C of continuous functions on the interval I: $0\leqq t\leqq 1$ which vanish at $t=0$, and $F(t,s,u)$ is continuous in (t,s,u) for (t,s) in the triangle Δ: $0\leqq s\leqq t\leqq 1$ and u in the infinite interval I: $-\infty<u<\infty$. The authors further assume that F satisfies a uniform Lipschitz condition with respect to u in every finite region: i.e., that there exists $M=M(U)$ such that

$$(2)\qquad |F(t,s,u_2)-F(t,s,u_1)|\leqq M|u_2-u_1| \text{ on } (t,s)\in\Delta,\quad |u_1|\leqq U,\ |u_2|\leqq U$$

for every finite positive U. Denote by T the transformation from x to y defined by (1); the hypotheses stated insure that T is a one-to-one transformation of C onto a subset Γ of C, and the inverse transformation defines $x(t)$ as a functional of y and t: $x(t)=T^{-1}(y|t)$. In two articles Cameron and the reviewer, and in a third article Cameron, B. W. Lindgren and the reviewer [Proc. Amer. Math. Soc. 3 (1952), 138–143; MR 13, 952] have evaluated the functional $T^{-1}(y|t)$ (for $\Gamma=C$) in terms of Wiener integrals. In the present paper the authors are able to strengthen the three types of results contained in these three papers by weakening and simplifying the hypotheses. The first of the three theorems being generalized gave an expression for $x(t)$ in terms of the weighted average under the assumption that the Lipschitz condition (2) holds in the whole infinite region, that is under the assumption that (2) holds with M independent of U. The authors' version of this theorem is considerably more general and, in fact, shows that the same expression for x holds with the condition that M be independent of U replaced by a mere order-of-magnitude condition.

The second of the three theorems being generalized gave an expression for $T^{-1}(y|t)$ in a series of Fourier-Hermite functionals. In generalizing this theorem the present authors are able to drop the stringent hypotheses relating to the behavior of F and certain of its derivatives on the diagonal. In order to do this, they make use of a recent version of a more general nonlinear transformation theorem than the one previously obtained by Cameron and the reviewer. The third of the three theorems which they generalize expresses the solution of the nonlinear equation as a limit of solutions of related linear problems. Here again they are able to obtain more general results.

W. T. Martin (Cambridge, Mass.).

See also: Tanimura, p. 629; Ahlfors, p. 657; Černyšenko, p. 665; Pogorzelski, p. 674; van Kampen, p. 690; Karp and Shmoys, p. 690; Visconti, p. 693; Nishijima, p. 694.

Calculus of Variations

Shiffman, Max. On surfaces of stationary area bounded by two circles, or convex curves, in parallel planes. Ann. of Math. (2) **63** (1956), 77–90.

Denote by S a minimal surface bounded by two plane curves Γ_1 and Γ_2 lying in parallel planes. The author proves the following. Theorem 1: If Γ_1, Γ_2 are circles, then every intersection of S by a plane parallel to the planes of Γ_1, Γ_2 is again a circle. Theorem 2: If Γ_1, Γ_2 are convex curves, every intersection of S by a plane parallel to the planes of Γ_1, Γ_2 is again a convex curve.

The proofs use the property of minimal surfaces, that the angle φ made by a tangent to a curve $z=$const. on the surface with the positively directed x-axis is harmonic in the surface metric. This fact, after a necessary discussion of the behavior of S at its boundaries, yields Theorem 2. To prove Theorem 1, the author introduces the function H conjugate harmonic to φ. In terms of isothermal coordinates u, $v=z$, H is related to the curvature K of a curve $z=$const. on S by the equation $K\cosh H=H_v$

Also the function $\beta(u, v)=K_u \cosh H$ satisfies the equation

$$(*) \qquad \beta_{vv}+\beta_{HH}\frac{2}{\cosh^2 H}\beta=0.$$

The author proves Theorem 2 by showing that a solution of (*) which vanishes on Γ_1 and Γ_2 either has constant sign on S or else vanishes identically.

As the author points out, Theorem 1 furnishes an example of boundary curves bounding more than one minimal surface all of which are known. In conjunction with work of Riemann [Oeuvres mathématiques, Gauthiers-Villars, Paris, 1898, esp. pp. 341–347] it permits the explicit determination of all minimal surfaces bounded by two circles in parallel planes. In addition to Theorems 1 and 2, the paper contains several related results and applications. Among these may be cited the theorem that if $x(u, v)$, $y(u, v)$ are two functions of class C'' defined over a region G and satisfying the relations $x_u^2+y_u^2=x_v^2+y_v^2+1$, $x_ux_v+y_uy_v=0$, then either x and y are functions of u alone or else the equations $x=x(u, v)$, $y=y(u, v)$, $z=v$ define a minimal surface. *R. Finn.*

Martin, A. D. A regular singular functional. Canad. J. Math. 8 (1956), 53–68.

Let J denote the functional

$$J[y; a, b]=\int_b^a [r(x)y'^2+2q(x)yy'+p(x)y^2]dx,$$

where $a<b$ and $r(x)$, $q(x)$, $p(x)$ are continuous in $(-\infty, +\infty)$. A function $y(x)$ is said to be F_t-admissible in $[a, b]$ if y is absolutely continuous, y' is L^2-integrable, and $y(a)=t$. As is well known, the regularity condition of J is expressed by $r(x)>0$, and then every solution of the Euler linear equation relative to J has a continuous derivative, and its unique solution, say $u(x)$, with $u(b)=0$, $u'(b)=-1$, generates a solution $z(x)=-r(x)u'(x)u^{-1}(x)-q(x)$ of the Riccati equation (R) $z'-(z+q(x))^2r^{-1}(x)+p(x)=0$ in any interval (a, b) where $u(x)\neq0$. In addition, the minimum of $J(y; t, b)$ in the class of all F_1-admissible functions in $[t, b]$ is $z(t)$. The present paper discusses the case where the condition $r(x)>0$ is removed so that the minimum of J need not be attained (neither Euler equation of J, nor equation (R) can be considered, and the points where $r(x)=0$ are singular). If $Z(t, b)$ denotes $\inf J[y; t, b]$ in the class above, then the properties of $Z(t, b)$ are discussed. If a Jacobi condition is satisfied (namely $c(a)\geqq b$ where $c(a)$ is the first conjugate at the right of a) then $Z(t, b)$ is continuous at the right everywhere, and of bounded variation. Also, if $R[x, z]$ denotes the first member of (R) if $r(x)\neq0$ and $-(z+q(x))^2$ if $r(x)=0$, then $R[x, Z(x, b)]=0$ for almost all x. Other properties of $Z(x, b)$ are given. [For previous studies see M. Morse and W. Leighton, Trans. Amer. Math. Soc. **40** (1936), 252–286; W. Leighton, ibid. **68** (1949), 253–274; W. Leighton and A. D. Martin, ibid. **78** (1955), 98–128; MR **11**, 603; **16**, 598.] *L. Cesari.*

Gass, S. I.; and Saaty, Thomas L. Parametric objective function. II. Generalization. J. Operations Res. Soc. Amer. 3 (1955), 395–401.

[For part I see same J. 2 (1954), 316–319; MR **16**, 51.] The problem discussed by the authors is that of determining the dependence of the structure of the basis of the solution of the problem of minimizing $\sum_{i=1}^n (a_i+\lambda_1b_i+\lambda_2c_i)x_i$ subject to $x_i\geqq0$, $\sum_{j=1}^n a_{ij}x_j=d_i$, upon the parameters λ_1 and λ_2. Two methods for treating this problem are presented. *R. Bellman* (Santa Monica, Calif.).

See also: Hartman and Wintner, p. 611; Bellman, p. 632; Souriau, p. 657.

Theory of Probability

★de Finetti, Bruno. La notion de „horizon bayesien". Colloque sur l'analyse statistique, Bruxelles, 1954, pp. 57–71. Georges Thone, Liège; Masson & Cie, Paris, 1955.

for order

This paper is primarily an exposition of ideas previously put forward by the author [see, e.g., Proc. 2nd Berkeley Symposium Math. Statist. Probability, 1950, Univ. of California Press, 1951, pp. 217–225; MR **13**, 851]. The problem is considered of reconciling orthodox (objectivistic, or "objective" for short) statistics with subjective (or personal) methods depending on inverse probability, and the view is expressed that the reconciliation should come through a multisubjective theory of probability. Wald's results are cited connecting "optimal" and minimax decision methods with Bayesian methods, but the minimax method is not regarded as especially reasonable for multisubjective decisions, and an initial (prior) distribution, if implicitly or explicitly used, should be one that is acceptable to all the people responsible for the decision. A "Bayesian horizon" is defined as a class of Bayesian decision methods (for a particular decision problem) that are unanimously preferred to any method outside the class. The properties of Bayesian horizons are not examined in detail in the present paper. It may be pointed out that most such properties should be applicable also to unisubjective probability, since a unisubjective probability (like a multisubjective one) cannot usually be judged to have a precise numerical value. [Cf. the relationship between Savage's "group minimax" method [The foundations of statistics, Wiley, New York, 1954, pp. 174–177; MR **16**, 147] and the reviewer's "type II" minimax" method [J. Roy. Statist. Soc. Ser. B. **14** (1952), 107–114, expressed more accurately in the proposal of the vote of thanks in a symposium on linear programming, ibid. **17** (1955), 194–196]. *I. J. Good* (Cheltenham).

Kemeny, John G. Fair bets and inductive probabilities. J. Symb. Logic **20** (1955), 263–273.

Let $C(H|E)$ be the credibility of H given E, or degree of confirmation (in Carnap's sense) of H by E. A betting system consists of choices of pairs (E, H), choice of a monetary stake for each pair, and choice whether to bet on or against H, when E is given. It is assumed that the odds of any bet are equal to $C(H|E)/(1-C(H|E))$. A set of credibilities is called fair if there is no betting system that guarantees a profit. It was shown by F. P. Ramsey [The foundations of mathematics and other logical essays, Paul, Trench, Trubner, London, 1931, pp. 156–198] and by B. de Finetti [Fund. Math. **17** (1931), 298–329; Ann. Inst. H. Poincaré **7** (1937), 1–68] that the condition of fairness implies that credibilities satisfy the usual axioms of probability. The author proves this in two pages and also that the axioms imply fairness. Shimony [J. Symb. Logic **20** (1955), 1–28; MR **16**, 1080] defined a 'strictly fair' body of credibilities as one for which no betting system exists both guaranteeing no loss and offering a possible gain. The author shows that strict fairness implies the 'controversial' axiom that what is almost impossible must be strictly impossible (in the real world). [Cf. H. Jeffreys's rule that any empirical

proposition must be formally capable of being accepted. Theory of probability, Oxford, 1939, p. 9; MR **1**, 151.]

There is an apparent suggestion in paragraph 1 that the axioms of credibility have no justification at all without the introduction of betting. The reviewer thinks, however, that a rough justification can be given in terms of 'equally probable cases', and also otherwise. [Cf., e.g., Probability and the weighing of evidence, Griffin, London, 1950, pp. 33, 105; MR **12**, 837; L. Janossy, Acta Phys. Acad. Sci. Hungar. **4** (1955), 333–349; MR **16**, 1127; J. Aczél, ibid. **4** (1955), 351–362; MR **16**, 1128.]

Credibilities could be replaced throughout by subjectivistic probabilities, with no change in the arguments.

I. J. Good (Cheltenham).

Lehman, R. Sherman. On confirmation and rational betting. J. Symb. Logic **20** (1955), 251–262.

This paper covers similar ground to the one reviewed above, although the work was carried out independently.

I. J. Good (Cheltenham).

Steyn, H. S. On discrete multivariate probability functions of hypergeometric type. Nederl. Akad. Wetensch. Proc. Ser. A. **58**=Indag. Math. **17** (1955), 588–595.

The multi-variable discrete probability functions introduced in an earlier paper [same Proc. **54** (1951), 23–30; MR **12**, 722] as coefficients of a multi-variable generating function of hypergeometric form are here shown to have the following properties: (i) the functions with any j of the k variables given fixed values are again of hypergeometric form; (ii) all marginal distributions are of hypergeometric form; and (iii) the limiting form for large values of the defining parameters is the multivariable normal. These results are illustrated for several special cases.

J. Riordan (New York N.Y.).

Larcher, M. P. L'emploi des fonctions factorielles dans les lois de probabilité d'Etienne Halphen. Publ. Inst. Statist. Univ. Paris **4** (1955), 38–39.

Various formulas for the distribution

$$f(z)=kz^{2\alpha-1}\exp(-z^2+xz),$$

where x, α are parameters. *G. Elfving.* (Helsingfors).

Sugiyama, Hiroshi. On the asymptotic behavior of $\sum p_m^2$ in case of certain probability distributions. II. Math. Japon. **3** (1955), 121–126.

Generalization of a previous result [Math. Japon. **2** (1952), 187–192; MR **14**, 993] to cover lattice distributions which have a finite variance and are infinitely divisible. *K. L. Chung* (Syracuse, N.Y.).

Grad, Arthur; and Solomon, Herbert. Distribution of quadratic forms and some applications. Ann. Math. Statist. **26** (1955), 464–477.

The paper deals with the distribution of $Q_k=\sum_1^k a_ix_i^2$, where $a_i>0$, $\sum a_i=1$, and the x_i's are independent and normal (0, 1). Densities and cumulative distribution functions are evaluated by means of Laplace transforms and represented as — rather complicated — real integrals. For $k=2$ and 3, tables are given and comparisons made to other approaches. *G. Elfving* (Helsingfors).

Ferguson, Thomas. On the existence of linear regression in linear structural relations. Univ. California Publ. Statist. **2** (1955), 143–165.

This paper is concerned with the following problem. Let $\mathbf{X}=(X_0, X_1, \cdots, X_n)$ be observable random variables related to the latent random variables $\boldsymbol{\xi}=(\xi_1, \cdots, \xi_s)$ and $\boldsymbol{\eta}=(\eta_0, \eta_1, \cdots, \eta_n)$ by the relation $\mathbf{X}=\mathbf{A}\boldsymbol{\xi}+\boldsymbol{\eta}$, where $\mathbf{A}$ is a constant matrix. If $\boldsymbol{\xi}$ and $\boldsymbol{\eta}$ are independent and the components of $\boldsymbol{\eta}$ are independent and at least one, η_1, say, is non-degenerate; under what circumstances is the regression of X_0 on $X_1, \cdots, X_n$ linear for some, or all, values of $\mathbf{A}$? Several results are obtained of which the following is typical and perhaps the most important: if $n>1$ a necessary and sufficient condition that the regression be linear for all $\mathbf{A}$ is that either (i) both $\boldsymbol{\xi}$ and $(\eta_1, \cdots, \eta_n)$ be normal, or (ii) $\eta_2, \cdots, \eta_n$ be degenerate and the logarithms of the characteristic functions of $\boldsymbol{\xi}$ and η_1 respectively be $-|\sum_{i=1}^s k_it_i|^\nu$ and $-K|t|^\nu$, where $1<\nu<2$, $k_i\neq0$ for all i, and $K>0$. The condition on $\boldsymbol{\xi}$ in (ii) may be expressed in terms of stable distributions. For the case $s=1$, $n\geq2$, aside from degeneracy (i) must be true even if the regression is to hold for only one $\mathbf{A}$. The case $n=1$ is considered separately, and the restriction that η_0 and η_1 be independent is dropped when s is also one. The connection with the problem of identifiability is discussed. *D. V. Lindley* (Cambridge, England).

Motoo, Minoru. Note on a relation between the distribution functions and characteristic functions. Ann. Inst. Statist. Math., Tokyo **6** (1955), 191–195.

The author proves the following Esseen-type theorem. Let $F(x)$ and $G(x)$ be two distribution functions and assume that both have moments of second order. Denote by $\varepsilon=\int_{-T}^{T}|f(t)-g(t)|\cdot|t|^{-1}dt$ $(T\geq1)$, and by $\delta=(1+\log T)/T$. Then $|F(x)-G(x)|\leq2\delta^{\frac{1}{2}}+\varepsilon$ for all x which do not belong to a certain set of measure not exceeding $4\delta^{\frac{1}{2}}$. The functions $f(t)$ and $g(t)$ are here the characteristic functions of $F(x)$ and $G(x)$ respectively. *E. Lukacs.*

Shapiro, J. M. A condition for existence of moments of infinitely divisible distributions. Canad. J. Math. **8** (1956), 69–71.

Let $F(x)$ be an infinitely divisible distribution and $\phi(t)$ be its characteristic function with the canonical representation of Lévy and Khintchine:

$$\phi(t)=\exp\left\{i\gamma t+\int_{-\infty}^{\infty}\left(e^{itu}-1-\frac{itu}{1+u^2}\right)\frac{1+u^2}{u^2}dG(u)\right\}.$$

It is proved that a necessary and sufficient condition for the $(2k)$th moment of $F(x)$ to be finite is that the $(2k)$th moment of $G(u)$ be finite, where k is any positive integer.

S. C. Moy (Detroit, Mich.).

Fisz, M. A limit theorem for a modified Bernoulli scheme. Studia Math. **15** (1955), 80–83.

Generalization of a previous result on multinomial distributions [Studia Math. **14** (1954), 272–275; MR **16**, 839] when the number of components tend to ∞.

K. L. Chung (Syracuse, N.Y.).

Krysicki, W. The limit theorem on terms of higher order on Bayes problem. Prace Mat. **1** (1955), 93–112. (Polish. Russian and English summaries)

Consider n independent trials with the same probability p of success and let p_0 denote the observable relative frequency of success, $0<p_0<1$. The probability p is treated as a random variable with a probability density $f(p)$ supposed continuous in [0, 1] and having a second derivative which, in a vicinity of p_0, is bounded. The subject of study is the a posteriori probability density, say $\varphi(p|p_0, n)$, of p given the fixed values of p_0 and n.

With reference to the familiar result of von Mises (and previously of S. Bernstein) to the effect that, as $n\to\infty$, the normed a posteriori density of p converges to the Gaussian density independently of f, the author deduces further terms of the asymptotic expansion of $\varphi(p|p_0, n)$ which happen to depend on f. In particular, for every $t\neq 0$ and with the exception of the case $p_0=\frac{1}{2}$, $f'(p_0)=0$,

$$\lim_{n\to\infty}\left\{\varphi\left(p_0+t\left(\frac{p_0(1-p_0)}{n}\right)^{\frac{1}{2}}\right)-\left(\frac{n}{2\pi p_0(1-p_0)}\right)^{\frac{1}{2}}e^{-\frac{1}{2}t^2}\right\}$$
$$=\left\{\frac{1-2p_0}{3p_0(1-p_0)}t^2+\frac{f'(p_0)}{f(p_0)}\right\}\frac{t}{(2\pi)^{\frac{1}{2}}}e^{-\frac{1}{2}t^2}\Bigg\}.$$

Another formula, depending upon the third derivative of f, is deduced for the previously excluded case $p_0=\frac{1}{2}$ and $f'(p_0)=0$. *J. Neyman* (Berkeley, Calif.).

Dugué, Daniel. Sur le second théorème limite du calcul des probabilités. C. R. Acad. Sci. Paris **242** (1956), 444–445.

Let $x_1, x_2, \cdots$ be independent random variables. The author states that for $\Pr(n^{-\frac{1}{2}}(x_1+\cdots+x_n)<x)$ to converge to a probability distribution whose characteristic function is $\exp(\psi(t))$, it is necessary and sufficient that .

$$(1)\qquad \lim_{m\uparrow+\infty}\ \limsup_{n\uparrow+\infty}\ \sum_{0<j\leq n}\Pr(|x_j|>n^{\frac{1}{2}}m)=0$$

and (2) there exist positive numbers, $m_1<m_2<\cdots<m_n\uparrow+\infty$, such that, for each real t,

$$\lim_{n\uparrow+\infty}\sum_{0<j\leq n}E(\exp(itn^{\frac{1}{2}}x_j-1, |x_j|<n^{\frac{1}{2}}m_n)=\psi(t).$$

Condition (1) is said to be equivalent to the requirement that

$$\sup_{\inf}\Pr(n^{-\frac{1}{2}}\max_{\min}(x_1, \cdots, x_n)<x)$$

be probability distributions. *H. P. McKean, Jr.*

Rosenblatt, M. A central limit theorem and a strong mixing condition. Proc. Nat. Acad. Sci. U.S.A. **42** (1956), 43–47.

A strong mixing condition for a sequence $X_1, X_2, \cdots$ of random variables is defined as the following: Let A, B be sets of the form $a_{kr}<X_{kr}\leq b_{kr}$, $r=1, 2, \cdots$. Let $I(A)$ be the smallest closed interval which contains all the indices of the random variables in terms of which A is defined. Let $d(A, B)$ be the distance between $I(A)$ and $I(B)$. The sequence $X_1, X_2, \cdots$ is said to satisfy the strong mixing condition if $|P(A\cap B)-P(A)P(B)|<f(d(A, b))$ where $f(n)\downarrow 0$ as $n\to\infty$. It is proved that if: 1) (a) $E|\sum_{j=a}^{b}X_j|^2\sim h(b-a)$ as $b-a\to\infty$, where $h(m)\uparrow\infty$ as $m\to\infty$; (b) $E|\sum_{j=a}^{b}X_j|^{2+\delta}=O(h(b-a))^{1+\frac{1}{2}\delta}$ as $b-a\to\infty$ for some $\delta>0$; 2) $EX_j\equiv 0$ and the strong mixing condition is satisfied; then k, p_n, q_n can be chosen such that p_n, q_n, $k\to\infty$ and $q_n/p_n\to 0$ as $n\to\infty$ and

$$(X_1+X_2+\cdots+X_n)/(kh(p_n))^{\frac{1}{2}}$$

is asymptotically normal. *S. C. Moy.*

Darling, D. A.; and Erdös, P. A limit theorem for the maximum of normalized sums of independent random variables. Duke Math. J. **23** (1956), 143–155.

"The main purpose of this paper is to prove the following theorem: Theorem. Let $X_1, X_2, \cdots$ be independent random variables with mean 0, variance 1, and a uniformly bounded third absolute moment. Put $S_k=X_1+X_2+\cdots+X_k$ and let

$$(1.1)\qquad U_n=\max_{1\leq k\leq n}\frac{S_k}{k^{\frac{1}{2}}}.$$

Then

$$\lim_{n\to\infty}\Pr\left\{U_n<(2\log\log n)^{\frac{1}{2}}+\frac{\log\log\log n}{2(2\log\log n)^{\frac{1}{2}}}+\frac{t}{(2\log\log n)^{\frac{1}{2}}}\right\}=$$
$$\exp(-e^{-t}(2/\pi)^{\frac{1}{2}}),\ -\infty<t<\infty.$$

"The idea behind the proof is quite simple though its execution is somewhat devious. We suppose first the X_i are Gaussian, independent, with mean 0 and variance 1. We then show there is a sequence $\{t_k\}$ and a stationary Gaussian stochastic process (the Uhlenbeck-Ornstein process) $X(t)$ such that the sequence $\{S_k/k^{\frac{1}{2}}\}$ ($k=1, 2, \cdots, n$) has the same joint distributions as $X(t_k)$ ($k=1, 2, \cdots, n$). It turns out, because certain machinery is available for stochastic processes, that the limiting distribution of $\max_{0<\tau<t}X(\tau)$ can be computed asymptotically when $t\to\infty$. And it is possible further to show that this limiting distribution is the same as the limiting distribution of $\max_{1\leq k\leq n}X(t_k)$ when $n\to\infty$, and so the same for U_n when $n\to\infty$. Next an application of the so-called invariance principle of Erdös-Kac will conclude the proof". (Quoted from the paper.) *J. Wolfowitz.*

★**Dugué, Daniel. Deux notions utiles en statistique mathématique: les ensembles aléatoires bornés „en loi" et la continuité fortement uniforme en probabilité.** Colloque sur l'analyse statistique, Bruxelles, 1954, pp. 133–141. Georges Thone, Liège; Masson & Cie, Paris, 1955.

Facile extension of Weierstrass approximation theorem, etc. to a stochastic process continuous in probability. *K. L. Chung* (Syracuse, N.Y.).

Theodorescu, Radu. Sur les relations caractéristiques des chaînes de Markoff continues de multiplicité p. Acad. R. P. Romîne. Bul. Şti. Secţ. Şti. Mat. Fiz. **7** (1955), 763–774. (Romanian. Russian and French summaries)

The author considers a continuous-parameter stochastic process with a finite state space. It is supposed that the process is a Markov process of multiplicity p, by which it is meant that, if $\mathbf{s}$ represents a vector $(s_1, \cdots, s_p)$, with $s_1\leq\cdots\leq s_p$, where $\mathbf{s}\leq\mathbf{t}$ means the corresponding inequality for the components, and if $\mathbf{i}$ represents a p-dimensional vector whose components are states, then the transition probability $P_{\mathbf{ij}}(\mathbf{s}, \mathbf{t})$ satisfies the conditions

$$P_{\mathbf{ik}}(\mathbf{r}, \mathbf{s})\geq 0,\ \sum_{\mathbf{k}}P_{\mathbf{ik}}(\mathbf{r}, \mathbf{s})=1,$$
$$P_{\mathbf{ik}}(\mathbf{s}, \mathbf{t})=\sum_{\mathbf{j}}P_{\mathbf{ij}}(\mathbf{s}, \mathbf{u})P_{\mathbf{jk}}(\mathbf{u}, \mathbf{t}),\ \text{if } \mathbf{s}\leq\mathbf{u}\leq\mathbf{t}.$$

For $p=1$ these are the standard relations valid for a Markov chain. Under appropriate regularity hypotheses there are then functions $Q_{\mathbf{jk}}^{h}(\mathbf{t})$ satisfying a compatibility condition, for which

$$\frac{\partial P_{\mathbf{ik}}(\mathbf{s}, \mathbf{t})}{\partial t_h}=\sum_{\mathbf{j}}P_{\mathbf{ij}}(\mathbf{s}, \mathbf{t})Q_{\mathbf{jk}}^{h}(\mathbf{t}),\ \mathbf{s}<\mathbf{t},\ h=1, \cdots, p.$$

This system of equations, with $\mathbf{t}$ variable and $\mathbf{s}$ fixed, and boundary condition $P_{\mathbf{ik}}(\mathbf{s}, \mathbf{s})=\delta_{\mathbf{ik}}$, determines the transition probabilities uniquely. There is a similar system corresponding to derivation with respect to s_h. Under

stronger regularity hypotheses the results remain valid for countable state spaces. [Note by reviewer. The probability background of the author's definition is not discussed. If each vector **s**, **t**, **u** above has all components the same, the relations become precisely those satisfied by the transition probabilities of an ordinary Markov chain. Thus it is not clear how general the author's case really is, nor what is its relation to the case $p=1$.]

J. L. Doob (Urbana, Ill.).

Theodorescu, Radu. Processus stochastiques de multiplicité p. Acad. R. P. Romîne. Bul. Şti. Secţ. Şti. Mat. Fiz. 7 (1955), 775–794. (Romanian. Russian and French summaries)

The author discusses chains of multiplicity p with a continuous parameter and a continuous state space, extending the results of another paper [see the preceding review] which treated countable state spaces. He obtains generalizations of the usual partial differential equations for the transition probabilities of Markov processes. The reviewer was unable to understand the only completely worked out example, apparently a kind of generalized Brownian motion, in which the stochastic process random variables at closely spaced time points are not nearly equal. *J. L. Doob* (Urbana, Ill.).

Homma, Tsuruchiyo. On a certain queuing process. Rep. Statist. Appl. Res. Un. Jap. Sci. Engrs. 4 (1955), 14–32.

"Recently various papers on a particular type of a single queuing system have been published. The queuing system is specified when we know (i) the input, (ii) the queue-discipline and (iii) the service-mechanism. Concerning the queuing-discipline it is supposed, in previous papers, that the customers form up into a single queue necessarily and that the customer at the head of the queue is served as soon as a server is free to attend to him. T. Kawata [same Rep. 3 (1955), 122–129; MR 16, 1035] has obtained a necessary and sufficient condition for ergodicity under a assumption that the customers join in a queue with given probabilities. Further he has supposed that the customers arrive at random and that the service-times follow the negative exponential law, and discussed by considering a differential equation about the associated pure birth process. In this paper, conditions about ergodic, null-recurrent and transient properties will be discussed under no special assumption about the service-time." (From the author's introduction.) *J. Wolfowitz.*

Getoor, R. K. The shift operator for non-stationary stochastic processes. Duke Math. J. 23 (1956), 175–187.

Let $(x(t)\colon t\geqq 0)$ be a complex stochastic process, let the second moments $E(|x(t)|^2)$ be $<+\infty$, and let the process be normal in the sense that the shift operators V_s: $x(t)\to x(t+s)$ have (possibly unbounded) normal extensions over some Hilbert space whose norm is an extension of the covariance $E(x(t)\overline{x(s)})$ which preserve the semi-group property $V_tV_s=V_{t+s}$. Using the spectral representation of these extended shift operators, the author characterises the covariance function, shows that the random variables $x(t)$ are transforms of a suitable random measure, and proves the appropriate (mean) ergodic theorem. When the extended shift operators are unitary, the process is (wide sense) stationary, and this is the most important special case. Other special cases have been studied by M. Loève [P. Lévy, Processus stochastiques et mouvement brownien, suivi d'une note de M. Loève, Gauthier-Villars, Paris, 1948; MR 10, 551].

Reviewer's note: in 1.2 on p. 183, read R^* for R and switch the entries in the inner products. The author wishes me to remark that Theorem 4E on p. 183 can be sharpened by cutting 1.18–21 and noticing that the expression in 1.17 is the projection of $x(0)$ on the maximal subspace invariant under the extended shift operators.

H. P. McKean, Jr. (Princeton, N.J.).

Kawata, Tatsuo. On the stochastic process of random noise. Kōdai Math. Sem. Rep. 7 (1955), 33–42.

Processes of the type $X_1(t)=\int_{-\infty}^{\infty}\Phi(t-s)dy(s)$ are considered, where $\Phi(t)$ is real on $(-\infty,\infty)$, and $y(t)$, $-\infty<t<\infty$, is a stochastic process whose sample functions are constant except for equidistributed jump increases U_i at instants generated by a Poisson process with parameter c. Some of the author's results are related to Rice's work [Bell System Tech. J. 23 (1944), 282–332; 24 (1945), 46–156; MR 6, 89, 233].

The normalized process $X(t)$ is defined as $X(t)=\int_{-\infty}^{\infty}\Phi(t-s)dY(s)$, $Y(t)=y(t)-ctE(U_i)$. The characteristic functions of $X(t)$, and $(X(t), X'(t))$ are obtained under certain assumptions, and used for a further study of $X(t)$. A typical result is Theorem 5: Suppose (i) $\Phi(t)$ is monotone for large $|t|>t_0$, $\Phi(t)\to 0$ as $|t|\to\infty$, (ii) $|\Phi(t)|\leqq A_r|t|^{-r}$ for every $r>0$, for large $|t|$, (iii) $|\Phi(t)-\Phi(t')|\leqq K|t-t'|^p$ $(p>0)$, (iv) U_i is a bounded random variable. Then $X(t)$ is continuous with probability 1. Theorem 7 gives a sufficient condition for $X(t)$ not to vanish over any interval, with probability 1. (The condition is in terms of $\Phi(t)$ and its Fourier transform.) Other results deal with the mean number of zeroes of $X(t)$ in a given interval, making use of a lemma of Kac [Bull. Amer. Math. Soc. 49 (1943), 314–320, 938; MR 4, 196; 5, 179]. *E. Reich.*

★**Lee, Y. W. On Wiener filters and predictors.** Proceedings of the symposium on information networks, New York, April, 1954, pp. 19–29. Polytechnic Institute of Brooklyn, Brooklyn, N.Y., 1955.

Let $f_m(t)$ be a message, $f_n(t)$ noise, $f_i(t)=f_m(t)+f_n(t)$ the input to a filter (predictor), $f_d(t)$ the desired output of the filter and $f_0(t)$ the actual output. Further, to specify the notation, let

$$\phi_{id}(\tau)=\lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T}f_i(t)f_d(t+\tau)dt.$$

The author considers a new criterion for a linear filter (predictor) consisting of the minimization of

$$\int_{-\infty}^{\infty}[\phi_{i0}(\tau)-\phi_{id}(\tau)]^2dt$$

instead of the familiar Wiener criterion of minimizing mean-square error. The computations for an optimum filter are carried out and comparisons are made between Wiener's results and these. *R. A. Leibler.*

★**Brillouin, L. Principe de néguentropie pour l'information.** Louis de Broglie, physicien et penseur, pp. 359–368. Editions Albin Michel, Paris, 1953. 870 francs.

An expository article on the relation between negentropy and Shannon's concept of information when both relate to a physical system. Several examples from physics illustrate the connection. [In formula (1) a minus sign is erroneously printed instead of an equal sign.]

R. A. Leibler (Washington, D.C.).

Schützenberger, Marcel Paul. Sur les problèmes de communications métriques. C. R. Acad. Sci. Paris **240** (1955), 724–726.

Defining a metric communication system as one whose loss function $L(\xi-X)$ is a definite quadratic form in $\xi-X$ vanishing only for $\xi=X$, the author considers the case $L(\xi-X)=\|\xi-X\|^2$. Here ξ and X are vectors representing, respectively, the state at the transmitter and the estimated state at the transmitter after the signal has been received. He shows that the value of X which minimizes expected $\|\xi-X\|^2$ is the conditionally expected ξ under the hypothesis of the received signal.

R. A. Leibler (Washington, D.C.).

Putnam, C. R. Temporal means and distribution functions. J. Soc. Indust. Appl. Math. **3** (1955), 137–141.

Soit $u(t)$ une fonction réelle, mesurable sur tout intervalle fini, telle que par exemple une des composantes de la vitesse d'un fluide turbulent. Soit

$$M_T(u)=T^{-1}\int_0^T u(t)dt$$

la moyenne temporelle de $u(t)$ sur l'intervalle T, $M(u)$ la limite, si elle existe, de $M_T(u)$ lorsque $T\to\infty$. On peut représenter $M_T(u)$ sous l'aspect d'une moyenne stochastique

$$M_T(u)=\int_{-\infty}^{\infty}\lambda d\sigma_T(\lambda)$$

en prenant pour $\sigma_T(\lambda)$ le quotient par T de la mesure de l'ensemble des points de $(0, t)$ pour lesquels $u(t)\leqq\lambda$. Si $|u(t)|$ est bornée, il existe une function de répartition $\sigma(u)$ telle que $\sigma_T(u)$ tende vers $\sigma(u)$ en tout point de continuité de $\sigma(u)$, et $M_T(u)$ vers $\int_{-\infty}^{\infty}\lambda d\sigma(\lambda)$ à condition que $\lim_{T\to\infty}T^{-1}\int_0^T u^k(t)dt$ existe pour tout k entier positif.

Cependant, on voit sur un exemple que $M_T(u)$ peut tendre vers une limite alors même que $\sigma(u)$ n'existe pas. Si $|u(t)|$ n'est pas bornée, il peut arriver que $\sigma_T(\lambda)$ tende vers une limite $\sigma(\lambda)$, et que $M_T(u)$ tende vers une limite différente de $\int_{-\infty}^{\infty}\lambda d\sigma(\lambda)$. On peut affirmer l'existence de $\sigma(\lambda)$ lorsque $u(t)=\int_{-\infty}^{\infty}\exp(i\lambda t)\,d\varrho(\lambda)$, où $\int_{-\infty}^{\infty}|d\varrho(\lambda)|<\infty$.

J. Bass (Paris).

Ramakrishnan, Alladi. Phenomenological interpretation of the integrals of a class of random functions. I, II. Nederl. Akad. Wetensch. Proc. Ser. A. **58**=Indag. Math. **17** (1955), 470–482, 634–645.

L'auteur considère un processus de Markoff homogène $x(\tau)$ défini par la densité de probabilité conditionnelle $\pi(x|x_0;\tau)$. Il se propose d'étudier des processus stochastiques représentés par des intégrales

$$(1)\quad y_n(t)=\phi_n(t)\int_0^t\phi_{n-1}(\tau_{n-1})d\tau_{n-1}\int_0^{\tau_{n-1}}\phi_{n-2}(\tau_{n-2})d\tau_{n-2}\cdots\int_0^{\tau_1}\phi_0(\tau_0)x(\tau_0)d\tau_0,$$

où ϕ_0, ϕ_1, $\cdots$, ϕ_n sont $n+1$ fonctions données.

I. Le „processus de base" est défini par la condition que $\pi(x|x_0;\Delta)$ soit équivalent, lorsque $\Delta\to 0$, à

$$R(x|x_0)\Delta+\delta(x-x_0)\left\{1-\Delta\int_x R(x|x_0)dx\right\}.$$

La trajectoire aléatoire correspondante est une courbe en gradins, qui subit un nombre fini de sauts dans un intervalle de temps donné.

Si $x(\tau)$ est un processus de base, on dira que $y_0(\tau)=\phi_0(\tau)x(\tau)$ est un processus d'ordre 0. La trajectoire aléatoire qui lui correspond a des discontinuités de première espèce: l'équation (1) ci-dessus définit un processus d'ordre n associé à $x(\tau)$. La courbe correspondante est continue d'ordre n associé à $x(\tau)$. La courbe correspondante est continue pour $n\geqq 1$. On vérifie que

$$\frac{dy_n(\tau)}{d\tau}=y_n(\tau)\frac{d}{d\tau}\log\phi_n(\tau)+\phi_n(\tau)y_{n-1}(\tau).$$

$y_n(\tau)$ est ainsi solution d'une équation différentielle linéaire d'ordre n contenant une fonction aléatoire $x(\tau)$. Dans le cas d'un processus du second ordre, cette équation s'écrit

$$\frac{d^2y_2}{dt^2}+\lambda_1\frac{dy_2}{dt}+\lambda_2y_2=\kappa x(t).$$

Sa résolution montre que $y_2(t)$ est la somme de deux processus du premier ordre.

II. Soient P_1, P_2, $\cdots$, P_n, B des points d'abscisses t_1, t_2, $\cdots$, t_n, t, les t_i étant aléatoires et distribués suivant la loi de Poisson. Si $n(\tau)$ est le nombre de points situés dans l'intervalle $AP=\tau$, l'intégrale $y(t)=\int_0^t n(\tau)d\tau$ se définit aisément, ainsi que l'intégrale $y(t)^*=\int_0^t n(t-\tau)d\tau$ de la „trajectoire inverse". Ces trajectoires sont différentes. Cependant $y(t)$ et $y(t)^*$ ont la même loi de probabilité. On dit que $y(t)$ et $y(t)^*$ sont des processus équivalents.

La densité de probabilité de $y(t)^*$ et la densité de probabilité conditionnelle de $y(t)$ pour $n(t)=n$ vérifient des équations fonctionnelles simples.

On peut étendre ces considérations à des intégrales de la forme

$$y(t)=\int_0^t n(\tau)\phi(t,\tau)d\tau.$$

Soient maintenant q_1, $\cdots$, q_n des variables aléatoires ayant même probabilité de densité $\psi(q)$. Au point P_n on associe le nombre $Q(t)=q_1q_2\cdots q_n$. $Q(t)$ est un „processus de base", discontinu aux points P_1, P_2, $\cdots$, P_n. On peut lui associer les intégrales

$$I(t)=\int_0^t Q(\tau)d\tau,\quad I^*(t)=\int_0^t Q(t-\tau)d\tau.$$

La densité de probabilité de $I^*(t)$ vérifie une équation fonctionnelle simple; l'intégrale $J(t)=\int_0^t d\tau\int_0^\tau Q(\tau')d\tau'$ représente le déplacement d'une particule brownienne soumise à une force proportionnelle à $Q(t)$. On peut lui associer quatre sortes de trajectoires directes ou inverses et calculer les moments de $J(t)$.

III. Si l'on pose d'une façon générale

$$Q_n(t)=\int_0^t Q_{n-1}(\tau)d\tau,\quad Q_1(t)=\int_0^t Q(\tau)d\tau,$$

on voit que

$$Q_n(t)=\int_0^t Q(\tau)\frac{(t-\tau)^{n-1}}{(n-1)!}d\tau.$$

La densité de probabilité $\pi(Q_n;t)$ de $Q_n(t)$ satisfait à l'équation fonctionnelle

$$\frac{\partial\pi(Q_n;t)}{\partial t}=-\lambda\pi(Q_n;t)+\lambda\int_q\pi\left(\frac{Q_n}{q};t\right)\psi(q)\frac{dq}{q}-\frac{t^{n-1}}{(n-1)!}\frac{\partial\pi(Q_n;t)}{\partial Q_n}.$$

D'une façon analogue, on peut étudier les processus définis par les intégrales

$$n_m(t)=\int_0^t n_{m-1}(\tau)d\tau,\quad n_1(t)=\int_0^t n(\tau)d\tau,$$

qui peuvent se représenter par la formule

$$n_m(t)=\int_0^t n(\tau)\frac{(t-\tau)^{m-1}}{(m-1)!}\,d\tau,$$

et auxquelles on peut associer les processus équivalents

$$\int_0^t n(t-\tau)\frac{\tau^{m-1}}{(m-1)!}\,d\tau.$$

J. Bass (Paris).

McMillan, Brockway. History of a problem. J. Soc. Indust. Appl. Math. 3 (1955), 119–128.

A number of interesting but loosely connected problems are more or less precisely formulated, and solved to various degrees. The original stimulation was provided by an engineering application.

Problem I. Let $f(t)$ belong to a class of functions having "small" $f''(t)$. Let $r(t)$, $r(t)=\pm 1$, $r(t)=$const for $n<t<n+1$ be an "approximation" of $f(t)$ subject to certain additional restrictions. To what extent can $f(t)$ be recovered by operations of the type $\int_{-\infty}^{t} K(t-t')r(t')dt'$?

Problem II. Given a stationary sequence of random variables f_n, does there exist a constant $S>0$, and a random sequence $r_n=\pm 1$, such that $\overline{z_m z_n}=0$, $\overline{z_m f_n}=0$, with $z_n=r_n-Sf_n$? If the f_n are bounded, the answer is yes.

Problem III. Describe the class U of covariance functions $R(n)$ of stationary sequences y_n, $y_n=\pm 1$. (Similarly UB for the case $|y_n|\leq 1$.) A necessary and sufficient condition for $R \in U$, which is analogous to definiteness, is given. It is sufficient for $R(n)$ to be of the form $(2/\pi)\text{arc sin } \varrho(n)$, $\varrho(n)$ positive definite, $\varrho(0)=1$. [Note: H. E. Scarf obtained the sufficient condition $R(n)=\int_0^{\frac{1}{2}} g(nx)dV(x)$, where $g(x)=1-4x$ $(0\leq x\leq \frac{1}{2})$, $g(-x)=g(x)=g(x+1)$, $V(x)\uparrow$ on $(0, \frac{1}{2})$, $V(0)=0$, $V(1)=1$.] *E. Reich.*

Arfwedson, G. Research in collective risk theory. II. Skand. Aktuarietidskr. 38 (1955), 37–100.

The first part of this paper was published in Skand. Aktuarietidskr. 37 (1954), 191–223 [MR 17, 275]. In the present final part the author gives first an asymptotic expression for the probability $Q(x, y)$ that the risk fund of initial value u will become negative at the end of the period x. An explicit formula for $F(x, u)$ is obtained by inverting its Fourier transform. Earlier results of the author [ibid. 36 (1953), 1–15; MR 15, 238] are again derived as special cases. Some asymptotic relations involving the functions $Q(x, u)$, $G(x, u)$, $\psi(u)$ are also derived. (For the notations see the reviews of the two papers cited above.) Most of the paper deals only with positive risk sums; two paragraphs at its end discuss briefly the negative and the mixed cases. *E. Lukacs.*

Segerdahl, C.-O. When does ruin occur in the collective theory of risk? Skand. Aktuarietidskr. 38 (1955), 22–36.

This is an exposition of some of the results contained in the author's dissertation [Uppsala, 1939]. The purpose of the present paper is to demonstrate the practical applicability of the theory by proving that the distribution of the first points of ruin is asymptotically normal. This follows from results contained in the author's earlier work but is not explicitly stated there. *E. Lukacs.*

Bierlein, Dietrich. Optimalmethoden für die Summenapproximation in Jecklins *F*-Methode. Bl. Deutsch. Ges. Versicherungsmath. 2 (1955), 291–352.

Uses game theory, and in particular the notion of an admissible minimax strategy, to formulate an "optimum" method of approximating the total reserves of an insurance portfolio. The method is applied to the estimation of a supposedly true total $\sum_x a_x t/(1-b_x t)$ by means of

$$\sum_x a_x t/\{1-\sum_x a_x b_x/\sum_x a_x\},$$

where x represents a typical policy in the portfolio that has been in force for t years. *H. L. Seal.*

See also: Taylor, p. 595; Sherman, p. 655.

Mathematical Statistics

★ **Fieller, E. C.; Lewis, T.; and Pearson, E. S. Correlated random normal deviates. 3,000 sets of deviates, each giving 9 random pairs with correlations** 0.1 (0.1) 0.9, **compiled from Herman Wold's Table of Random Normal Deviates (Tract no. XXV).** Cambridge, at the University Press, 1955. xv+60 pp. $2.00.

Three thousand observations of the random vector $X=(X_0, X_1, \cdots, X_9)$ are given. The distribution of X is multivariate normal with mean value 0 and covariance matrix $|\sigma_{ij}|$, where $\sigma_{ii}=1$, $\sigma_{0j}=j/10$ and $\sigma_{kj}=kj/100$ for $i=0, \cdots, 9$; $j\neq k$; and j, $k=1, \cdots, 9$. The method of construction is indicated and tests of randomness give satisfactory results. Examples of uses are absent. *I. R. Savage* (Stanford, Calif.).

Armsen, P. Tables for significance tests of 2×2 contingency tables. Biometrika 42 (1955), 494–511.

Compact tables, based on the exact hypergeometric series, are given for making one- and two-tailed tests of significance at the 5% and 1% levels in 2×2 contingency tables containing up to 50 observation. Alternative rules for defining the region of rejection in the case of two-tailed tests are discussed before the rule used in constructing the tables is adopted. *W. G. Cochran.*

Pieruschka, E. Die statistische Verteilung für das Auftreten des zweiten, dritten, oder allgemein *n*-ten Fehlers eines Geräts. Z. Angew. Math. Mech. 35 (1955), 470–473.

Bennett, B. M. On the cumulants of the logarithmic generalized variance and variance ratio. Skand. Aktuarietidskr. 38 (1955), 17–21.

Let $|V|$ be the determinant of the sample variance-covariance matrix based on n independant observations on a p-variate multivariate normal population. Let $|V_1|$, $|V_2|$ be two such independent determinants based on n_1, n_2 observations, respectively, on two possibly different p-variate normal populations (p the same for both). The moment generating function and cumulant generating functions for log $|V|$ and log $(|V_1|/|V_2|)$ are obtained in a closed-form expression. *M. Dwass* (Stanford, Calif.).

Hofmann, Martin. Über zusammengesetzte Poisson-Prozesse und ihre Anwendungen in der Unfallversicherung. Mitt. Verein. Schweiz. Versich.-Math. 55 (1955), 499–575.

Develops two discrete bivariate probability distributions, the marginal (univariate) distributions of which are negative binomial distributions. Both belong to the family of bivariate compound Poisson distributions [Consael, Acad. Roy. Belg. Cl. Sci. Mém. Coll. in 8°

27 (1952), no. 6; MR **15**, 138]. The second of them is applied successfully to the number of work-connected and other accidents suffered by 1196 male workers during the nine years 1944–52. The author argues that the observations justify the view that the individual has a varying accident-proneness at the outset. *H. L. Seal.*

Bennett, B. M. On the joint distribution of the mean and standard deviation. Ann. Inst. Statist. Math., Tokyo 7 (1955), 63–66.

The author derives a recursion formula for the joint frequency function of the sample mean and sample variance from a general population. It is assumed that the population distribution function is absolutely continuous and that the corresponding frequency function has no zeros. *E. Lukacs* (Washington, D.C.).

Banerjee, D. P. A note on the distribution of the ratio of sample standard deviations in random samples of any size from a bi-variate correlated normal population. J. Indian Soc. Agric. Statist. 6 (1954), 93–100.

The exact distribution of the ratio of sample standard deviations in random samples of size N from a bi-variate correlated normal distribution is found in terms of Student's t, when the population variances are equal. Probability points for this ratio are given for: 1. correlations of 0(.1).9; 2. sample sizes of 3(1)30; 3. probabilities of .8, .9, .95, and .99. *I. R. Savage* (Stanford, Calif.).

Tukey, John W. Interpolations and approximations related to the normal range. Biometrika 42 (1955), 480–485.

Various characteristics of normal range (e.g., percentage points, expectation, reciprocal standard deviation) behave asymtotically like the square root of $a \log (bn+c)$, where n is the sample size, and a, b, c appropriate constants. This fact can be exploited for interpolating the said characteristics between standard values of n, e.g., 20, 100, 1000. *G. Elfving* (Helsingfors).

Tukey, John W. Some selected quick and easy methods of statistical analysis. Trans. New York Acad. Sci. (2) 16 (1953), 88–97.

Exemplification of various "quick"methods, mainly based on ranges. *G. Elfving* (Helsingfors).

Oderfeld, J.; and Stanisławski, J. On a certain acceptance regulation. Zastos. Mat. 2 (1955), 328–340. (Polish. Russian and English summaries)

Oderfeld, J.; and Wiśniewski, K. Sampling in control of quality taking account of errors made by the inspectors. Zastos. Mat. 2 (1955), 312–327. (Polish. Russian and English summaries)

Ikeda, Sadao. On the estimation of the quality of a group of lots by the single sampling inspection in destructive case. Osaka Math. J. 7 (1955), 131–156.

Some of the results due to Girshick, Mosteller, and Savage [Ann. Math. Statist. **17** (1946), 13–23; MR **8**, 477], Wolfowitz [ibid. **17** (1946), 489–493; MR **8**, 477], and Savage [ibid. **18** (1947), 295–297; MR **9**, 152] for unbiased estimates in binomial sampling are reproduced. The author also includes unbiased estimates in hypergeometric sampling. Following Kolmogorov [Izv. Akad. Nauk SSSR. Ser. Math. **14** (1950), 303–326; MR **12**, 116; **15**, 452] the author presents some approximations for estimating the quality of a group of lots by single sampling inspection. *M. Muller* (Ithaca, N.Y.).

Cohen, A. Clifford, Jr. Restriction and selection in samples from bivariate normal distributions. J. Amer. Statist. Assoc. 50 (1955), 884–893.

A concise exposition of recent results obtained by the author and by others on maximum likelihood estimation of parameters in situations indicated in the title. The explicit discussion of some special cases is new. An illustrative numerical example is presented.
Z. W. Birnbaum (Seattle, Wash.).

Bhattacharyya, M. N. Estimation from censored bivariate samples. J. Indian Soc. Agric. Statist. 6 (1954), 83–92.

The maximum-likelihood equations (m.l.e.) are derived for estimating the means and standard deviations of an uncorrelated bivariate normal distribution from a sample doubly censored on both variables. The information matrix is derived. An iterative computational technique for solving the m.l.e. is given and illustrated. This paper is a partial generalization of material summarized by Cohen [see the paper reviewed above]. *I. R. Savage.*

Deemer, Walter, L. Jr.; and Votaw, David F., Jr. Estimation of parameters of truncated or censored exponential distributions. Ann. Math. Statist. 26 (1955), 498–504.

For a random variable X with the probability density $h(X)=ce^{-cX}$, $X>0$, maximum likelihood estimators of c are obtained (A) when $h(X)$ is censored and (B) when it is truncated at a known value X_0. In case (A) the solution of the likelihood equation is easily computed; in case (B) an auxiliary table given in the paper simplifies the numerical solution. Asymptotic variances of the two estimators are derived, confidence intervals for c are obtained for large sample sizes, and the results of a sampling experiment are reported which suggest that the limiting normal distributions of the estimators yield a fairly good approximation when $cX_0 \leqq 1$, and sample size $\geqq 100$. *Z. W. Birnbaum* (Seattle, Wash.).

den Broeder, George Gerard, Jr. On parameter estimation for truncated Pearson type III distributions. Ann. Math. Statist. 26 (1955), 659–663.

The author obtains maximum-likelihood estimators for α in the p.d.f.

$$\varphi(t, \alpha)=\Gamma^{-1}(p)\alpha^{p}t^{p-1}e^{-\alpha t}, \quad 0\leqq t, \quad 0<\alpha, \quad 0<p,$$

when p is known, and there has been truncation on the right or on the left. He treats both the cases of known and unknown numbers of truncated individuals and compares the corresponding losses in information. *H. A. David.*

Suzuki, Yukio. Note on the Neyman-Pearson's fundamental lemma. Ann. Inst. Statist. Math., Tokyo 6 (1955), 197–211.

Let $\{f_i\}$ $(i=0, 1, \cdots)$ be a sequence of probability densities on a Euclidean space R, let $\mathcal{S}$ denote the class of Borel sets S with $\int_S f_i dx=c_i$ $(i=1, \cdots)$, and let $\mathcal{S}_0$ denote the class of $S \in \mathcal{S}$ for which $\int_S f_0 dx$ is maximized. If there is a sequence $\{k_i\}$ of real numbers such that $\sum_1^\infty k_i f_i(x)$ is finite almost everywhere and $\sum_1^n k_i f_i(x) \leqq G(x)$ for all x, n, where G is integrable, and if $S \in \mathcal{S}$ satisfies $f_0 \geqq \sum_1^\infty k_i f_i$ for $x \in S$, $f_0 \leqq \sum_1^\infty k_i f_i$ for $x \notin S$, then $S \in \mathcal{S}_0$. Under further assumptions, the condition is necessary also. The results

extend a theorem of Dantzig and Wald [Ann. Math. Statist. 22 (1951), 87–93; MR 12, 622] for a finite set of densities. *D. Blackwell* (Berkeley, Calif.).

Basu, D. On statistics independent of a complete sufficient statistic. Sankyā 15 (1955), 377–380.

Let $\{P_\theta\}$, θ in Ω, be probability measures on a sample space, and let the statistic T be sufficient. Theorem 1: If the statistics T_1, T are statistically independent, then the distribution of T_1 does not depend on θ. Theorem 2: If T is a boundedly complete sufficient statistic and the distribution of T_1 does not depend on θ, then T_1 is statistically independent of T. The proof follows directly from the definitions of sufficiency and completeness. Some applications are given. *M. Dwass* (Stanford, Calif.).

van Eeden, Constance. Maximum likelihood estimation of ordered probabilities. Math. Centrum Amsterdam. Statist. Afdeling Rep. **S 188 (VP 5)** (1956), 8 pp.

Let k ($k\geqq 2$) independent series of Bernoulli trials be made, where the ith series consists of n_i Bernoulli trials and where a_i is the number of observed successes. A procedure is given for finding maximum-likelihood estimates of the true (but unknown) success ratios π_i ($i=1, \cdots, k$) subject to the a priori restriction that $\pi_1\leqq\pi_2\leqq\cdots\leqq\pi_k$. A generalization to the case where the π_i ($i=1, \cdots, k$) satisfy other sorts of inequalities is considered. Numerical examples illustrating the procedure are given. The paper by Ayer, Brunk, Ewing, Reid and Silverman [Ann. Math. Statist. 26 (1955), 641–647; MR 17, 504] gives the same estimates for the case where $\pi_1\leqq\pi_2\leqq\cdots\leqq\pi_k$ and furthermore proves consistency of the maximum-likelihood estimates, which the paper being reviewed does not do. *B. Epstein* (Stanford, Calif.).

Gani, J. Some theorems and sufficiency conditions for the maximum-likelihood estimator of an unknown parameter in a simple Markov chain. Biometrika 42 (1955), 342–359.

"The paper begins with proofs of the usual theorems for the optimum properties of the maximum-likelihood estimator of an unknown parameter θ which defines the transition probabilities $p_{ij}(\theta)$ of a simple ergodic Markov chain. The paper proceeds to establish the form of the transition probabilities $p_{ij}(\theta)$ which admit a sufficient estimator of θ. The paper closes with an examination of possible forms for the matrix p of transition probabilities $p_{ij}(\theta)$, and these are illustrated with simple examples for Markov chains with two and three states." (From the author's summary.) *J. Wolfowitz* (Ithaca, N.Y.).

Elfving, G. Geometric allocation theory. Skand. Aktuarietidskr. 37 (1954), 170–190 (1955).

Let $X_1, \cdots, X_n$ be independent k vectors such that $X_i=A_i\alpha+\xi_i$, where A_i is a known $m\times k$ matrix, α is an unknown k-dimensional parameter, and ξ_i is a k-dimensional error vector with $E\xi_i=0$, $E(\xi_i\xi_i')=\Lambda_i$ non-singular. For a fixed class C of pairs $c=(A, \Lambda)$, denote by $R(c)$ the class of row vectors r such that the least-squares estimate of $r\alpha$ has variance $\leqq 1$, when $A_i\equiv A$, $\Lambda_i\equiv\Lambda$, and by R the corresponding class when each $(A_i, \Lambda_i)\in C$. Each $R(c)$ is convex, and R is contained in the convex hull of $\bigcup_c R(c)$, so that, for large sample optimum design of an experiment to estimate a given linear function $r\alpha$, at most k different experiments (c-points) need be used. *D. Blackwell* (Berkeley, Calif.).

Walsh, John E. Bounded significance level tests for comparing quantiles of two possibly different continuous populations. Ann. Inst. Statist. Math., Tokyo 6 (1955), 213–222.

Let F, G be continuous c.d.f.'s unknown to the experimenter, α and β preassigned constants between 0 and 1, and a and b constants such that $F(a)=\alpha$ and $G(b)=\beta$. On the basis of independent observations on random variables distributed as F and G, approximate confidence intervals are given for estimating $a-b$. There is some study of the "asymptotic efficiency" of the procedure. *M. Dwass.*

Ghosh, M. N. Simultaneous tests of linear hypotheses. Biometrika 42 (1955), 441–449.

"A very common situation in the analysis of variance of survey data, where the investigator is not able to put his experimental data in the framework of a design, planned in advance, is that the estimates of the parameters are correlated in various ways and the analysis of variance becomes cumbersome. Even for the analysis of variance of a two-way classification with unequal class frequencies one meets with this difficulty. We shall be concerned here with the analysis of such data and the test of significance for groups of parameters representing different aspects of the problem, in a concrete example." (From the author's summary.) *J. Wolfowitz.*

Sobel, Milton. Statistical techniques for reducing the experiment time in reliability studies. Bell System Tech. J. 35 (1956), 179–202.

"Given two or more processes, the units from which fail in accordance with an exponential or delayed exponential law, the problem is to select the particular process with the smallest failure rate. It is assumed that there is a common guarantee period of zero or positive duration during which no failures occur. This guarantee period may be known or unknown. It is desired to accomplish the above goal in as short a time as possible without invalidating certain predetermined probability specifications. Three statistical techniques are considered for reducing the average experiment time needed to reach a decision.

"1. One technique is to increase the initial number of units put on test. This technique will substantially shorten the average experiment time. Its effect on the probability of a correct selection is generally negligible and in some cases there is no effect.

"2. Another technique is to replace each failure immediately by a new unit from the same process. This replacement technique adds to the bookkeeping of the test, but if any of the population variances is large (say in comparison with the guarantee period) then this technique will result in a substantial saving in the average experiment time.

"3. A third technique is to use an appropriate sequential procedure. In many problems the sequential procedure results in a smaller average experiment time than the best non-sequential procedure regardless of the true failure rates. The amount of saving depends principally on the "distance" between the smallest and second smallest failure rates.

"For the special case of two processes, tables are given to show the probability of a correct selection and the average experiment time for each of three types of procedures.

"Numerical estimates of the relative efficiency of the procedures are given by computing the ratio of the

average experiment time for two procedures of different type with the same initial sample size and satisfying the same probability specification." (From the author's summary.) Examples illustrate the theory.

L. A. Aroian (Culver City, Calif.).

Dalcher, Andreas. Statistische Schätzungen mit Quantilen. Mitt. Verein. Schweiz. Versich.-Math. **55** (1955), 475–498.

A parameter ϑ of a univariate continuous probability distribution $f(x)$ is written as $\vartheta=\sum_{i=1}^{n} b_i\xi_i$, where

$$\int_{-\infty}^{\xi_i} f(x)dx=n^{-1}\left\{\sum_{j=0}^{i-1}(a_j+1)-\tfrac{1}{2}\right\},$$

and a_j, k and n are arbitrary positive integers. The proposed unbiased estimate of ϑ from a random sample of n x's is $T=\sum_{i=1}^{k} b_i y_i$, where y_i is the value of x in the $\sum_{j=0}^{i-1}(a_j+1)$-th rank and the a's and b's are determined to minimize the asymptotic variance of T. The general method is developed and numerical results (with k-values up to 5) are provided for the parameter of the normal, Cauchy and exponential distributions. *H. L. Seal.*

Gervaise, Anne-Marie. Risque d'erreur dans un test d'hypothèse appliqué à un paramètre aléatoire, moyenne de k paramètres indépendants lorsque la taille de l'échantillon varie avec k. C. R. Acad. Sci. Paris **242** (1956), 729–730.

Doornbos, R.; and Prins, H. J. A slippage test for a set of Gamma-variates. Math. Centrum Amsterdam. Statist. Afdeling Rep. **S 187 (VP 4)** (1956), 10 pp.

Modifications of a procedure of Cochran [Ann. Eugenics 11 (1941), 47–52; MR 3, 171] are presented to cover the situations of unequal sample sizes and slippage to the left. Approximate distributions are given for the test statistics under both the null and alternative states.

I. R. Savage (Stanford, Calif.).

Williams, E. J. Tests of significance for concurrent regression lines. Biometrika **40** (1953), 297–305.

Suppose there are several sets of observations on bivariate data and the linear regression for each set has been calculated. Two problems are considered: (1) a test of the null hypothesis that the regression lines are concurrent; (2) determination of fiducial limits for the value of the independent variable of the point of concurrence. Procedures developed by the author for the interpretation of interactions in factorial experiments [Biometrika 39 (1952), 65–81; MR 14, 300] are applied to the solution of these problems first discussed by Tocher [ibid. 39 (1952), 109–117; MR 13, 963]. Two numerical examples are given, one of which is a re-examination of Tocher's data. *S. Kullback* (Washington, D.C.).

Thompson, W. A., Jr. The relative size of the inter- and intra-block error in an incomplete block design. Biometrics **11** (1955), 406–426.

The author considers the analysis of BIB based on the variance-components model, where the block effects are assumed to be normally and independently distributed random variables. He is particularly interested in the ratio of the interblock variance to the intrablock variance and develops tests of hypotheses and confidence intervals for this statistic. The methods are illustrated by an example where this ratio is of practical importance. The theory as well as the computational procedures are treated in detail. The methods are also extended to certain classes of partially balanced designs. *H. B. Mann.*

Ward, G. C.; and Dick, I. D. Non-additivity in randomized block designs and balanced incomplete block designs. New Zealand J. Sci. Tech. Sect. B. **33** (1952), 430–435.

Let t_i be the treatment effect and b_j the block effect producing the yield y_{ij}. The authors assume that $y_{ij}=m+t_i+b_j+nt_ib_j+e_{ij}$, where m is the mean n an unknown parameter and the e_{ij} are normally and independently distributed random variables all with the same distribution. The author proposes an iterative method for the solution of the resulting least-square equations in the case of the designs in the title. *H. B. Mann.*

Dick, I. D.; and Whittle, P. Contributions to the statistical design of identical twin experiments. New Zealand J. Sci. Tech. Sect. B. **33** (1951), 145–172.

Part I is a general discussion of block experiments with two plots per block, with possible replication in time and with different observers. The relative efficiencies are examined of balanced incomplete blocks, quasi-factorials, and designs in which one particular treatment appears as a control in each block. The least-squares solution for the last design is given. Part II gives the least-squares solution for time-replicated non-orthogonal experiments with equal block size and equal replication for each treatment. Part III deals with estimation of components of variance in the same situation. Part IV gives the treatment for one or two missing readings in balanced incomplete block or quasi-factorial designs with two plots per block.

P. Armitage (London).

Williams, E. J. Use of scores for the analysis of association in contingency tables. Biometrika **39** (1952), 274–289.

The problem considered is that of interpreting the association known to exist in a contingency table. In particular, the methods discussed depend on the assignment of a single value or score to each class. Three cases are considered: (1) Scores for both classifications given in advance; (2) scores for one classification given in advance and the score for the other classification estimated so as to maximize the correlation between the two sets; (3) scores for both classifications estimated so as to maximize the correlation between the two sets. Tests of significance under appropriate assumptions are discussed in detail.

S. Kullback (Washington, D.C.).

Mosetti, Ferruccio. Su di un metodo di analisi delle periodicità. Ann Geofis. **8** (1955), 331–349.

On the moving summation $y_t^*=\sum_0^n a_k y_{t-k}$ by which a harmonic sum $\sum_0^m \sin(\varphi_i+\lambda_i t)$ can be eliminated in a given time-series y_t. *H. Wold* (Uppsala).

See also: Seiden, p. 571; Thompson, p. 572; Primrose, p. 572; Roy, p. 572; Sprott, p. 572; Das, p. 572; Krishna Iyer, p. 572; de Finetti, p. 633; Quartey, p. 665; Fürst, p. 665.

Theory of Games, Mathematical Economics

Bellman, R.; Glicksberg, I.; and Gross, O. On the optimal inventory equation. Management Sci. **2** (1955), 83–104.

This paper considers the "optimal inventory problem",

which is a particular case of the general problem of inventory ordering in the face of an uncertain future demand. The problem is formulated as follows: To determine the optimal inventory ordering policy of a firm that will minimize mathematical expectation of costs. Costs are taken to consist of two parts: initial ordering costs+penalty depletion costs or ordering to meet an excess of demand over supply. [For the origin of this problem, see Arrow, Harris, and Marschak, Econometrica **19** (1951), 250–272; MR **13**, 368; Dvoretzky, Kiefer and Wolfowitz, ibid. **20** (1952), 187–222, 450–466; MR **13**, 856; **14**, 301.] A new proof of the existence and uniqueness theorems for the class of functional equations which arise is given, utilizing the Picard method of successive approximations. It is shown that when all ordering costs are directly proportional to the amounts ordered, the optimal policy is characterized by the principle of constant inventory level. The analysis is tentatively generalized in various directions, multi-dimensional study, non-proportional penalty costs, etc.

Section 2.1 has been misplaced and is to be inserted on p. 85 before formula (1). On p. 98, theorem 3, equation 7(a) should read "for $0\leq x_i\leq \bar{x}_i$, $y_i=\bar{x}_i$". *S. A. O. Thore.*

Balk, W. Contribution à la théorie de la distribution des revenus. Verzekerings-Arch. Actuarieel Bijvoegsel **33** (1956), 1*–18*.

If $f(u|x, t)$ represents the continuous density function of incomes received by individuals aged x at time t, the relation

$$f(u|x, t)=g\{\psi(u, x, t)\}\frac{\partial\psi(u, x, t)}{\partial u}$$

is interpretable in terms of the future incomes of these individuals [Kaiser, Mitt. Verein. Schweiz. Versich.-Math. **55** (1955), 305–335; MR **16**, 1131] and may be obtained as the solution of

$$\frac{\partial f(u|x, t)}{\partial t}+\frac{\partial f(u|x, t)}{\partial x}+\frac{\partial uk(u, x, t)f(u|x, t)}{\partial u}=0.$$

The author substitutes $g(u, x, t)f(u|x, t)$ for the right-hand side of this partial differential equation and derives a generalized relation for $f(u|x, t)$. *H. L. Seal.*

Leuenberger, Franz. Zur mathematischen Theorie der Einkommensverteilung in Abhängigkeit von Alter und Zeit. Mitt. Verein. Schweiz. Versich.-Math. **55** (1955), 577–615.

In social insurance it is often necessary to study the distribution of a population characterized by three variables: chronological time, income and age. A mathematical model for such studies was constructed by E. Kaiser [same Mitt. **50** (1950), 249–335; MR **12**, 516]. The author extends Kaiser's investigations by proposing modifications of Kaiser's hypothesis A. He postulates first (hypothesis A_1) that at any time the conditional frequency function $f(u|x)$ of incomes u, given the age x, depends on x only by means of a scale and a location parameter. Moreover, these parameters are connected by the assumption, already made by Kaiser, that the dependence of the mean income on the age is known. An alternative proposal (hypothesis A_2) implies that all $f(u|x)$ belongs to a one-parameter family of frequency functions where the parameter is a function of x. The consequences of both hypotheses are studied. They are also applied to special income distributions such as Pareto's distribution and the Gamma distribution.

E. Lukacs (Washington, D.C.).

See also: Bellman, p. 632.

TOPOLOGICAL ALGEBRAIC STRUCTURES

Numakura, Katsumi. Theory of compact rings. Math. J. Okayama Univ. **5** (1955), 79–93.

A theorem of Asano [J. Math. Soc. Japan 3 (1951), 82–90; MR **13**, 313] reads as follows: if A is a Noetherian commutative ring with unit such that for any maximal ideal M there is no ideal between M and M^2, then A is a direct sum of a finite number of Dedekind domains and principal-ideal rings with descending-chain condition. The author's main objective is to prove an analogous result for a compact ring A. He allows A to be non-commutative, assuming that any two maximal open left (or right) ideals commute and that there are no one-sided ideals between an open maximal ideal and its square. In the conclusion an infinite direct sum may appear, and the summands are either finite principal-ideal rings, or maximal orders in totally disconnected locally compact division rings. There are a number of other related theorems. *I. Kaplansky.*

Kowalsky, Hans-Joachim. Stonesche Körper und ein Überdeckungssatz. Math. Nachr. **14** (1955), 57–64.

A subset S of a topological division ring K is said to be bounded if for every neighborhood N of 0, there is a neighborhood P of 0 such that PS and SP are contained in N. The topological division ring K is said to be of type V if every subset S of K that is bounded away from 0 (i.e., there is a neighborhood N of 0 such that $N\cap S$ is empty) is such that S^{-1} is bounded. A function with values in K is said to be bounded if its range is a bounded subset of K. Let $C^*(X, K)$ denote the ring of all bounded continuous functions on a topological space X with values in K. If for every topological space X, and every maximal (two-sided) ideal M of $C^*(X, K)$, the residue class ring $C^*(X, A)$ is isomorphic to K, then K is called a Stone division ring.

Goldhaber and Wolk [Duke Math. J. **21** (1954), 565–569; MR **15**, 968] have shown that if K satisfies the first axiom of countability and is of type V, then K is a Stone division ring only if it is locally compact. They asked if every locally compact division ring is a Stone-division ring, and if every maximal ideal of $C^*(X, A)$ is of a certain form (too complicated to be given here). The author generalizes this theorem by removing the countability assumption, and answers the question yes, respectively no.

Reviewer's comments: 1. Ellen Correl and the reviewer have also answered these questions — see the review of the Goldhaber-Wolk paper and Proc. Amer. Math. Soc. 7 (1956), 194–198. 2. I am puzzled by the author's remark on p. 63 that not every locally compact division ring is of type V since it is well known [cf., e.g., I. Kaplansky, Bull. Amer. Math. Soc. **54** (1948), 809–826; MR **10**, 179] that every locally compact division ring admits a valuation preserving its topology. *M. Henriksen*

Barot, Jiří. Remark on inverse elements in topological rings. Časopis Pěst. Mat. **80** (1955), 241–243. (Czech)

Let R be a ring with unit e that is also an L^*-space in the sense of Fréchet. Suppose that addition and subtraction are continuous in R in both variables and that multipli-

cation by a fixed element is continuous in one variable. Theorem: If $x \in R$, $x \neq 0$, and $\lim_{n\to\infty} \sum_{i=0}^{n} x^i$ exists, then $(e-x)^{-1}$ exists. *E. Hewitt* (Princeton, N.J.).

Ballier, Friedhorst. Über lineartopologische Algebren. J. Reine Angew. Math. **195** (1956), 42–75 (1955).

A vector space is linearly topologized if it has subspace neighborhoods of 0; an algebra is linearly topologized if it has subalgebra neighborhoods of 0. In § 1 the author presents basic facts on linearly topological algebras. In § 2 he adds the hypothesis of linear compactness, which among other things gives rise to ideal neigborhoods of 0. Various results on the radical are given, and it is proved that semi-simple linearly compact algebras are complete direct sums of finite-dimensional simple algebras. Some of the results are extended to the locally linearly compact case. In § 3 it is postulated that there exists a complete set of continuous homomorphisms from the algebra to the base field; one can then prove results analogous to those which hold for Banach algebras and locally convex algebras. The author notes that there is some overlapping with a paper of Zelinsky [Amer. J. Math. **75** (1953), 79–90; MR **14**, 532]. *I. Kaplansky* (Chicago, Ill.).

Topological Groups

Wallace, A. D. One-dimensional homogeneous clans are groups. Nederl. Akad. Wetensch. Proc. Ser. A. **58** = Indag. Math. **17** (1955), 578–580.

A clan is a compact connected Hausdorff semigroup with an identity element. A space X is homogeneous if for any two points x and y in X there is a homeomorphism of X onto X taking x into y. The dimension used here is the Lebesgue covering dimension although many of the proofs make use of codimension [See H. Cohen, Duke Math J. **21** (1954), 209–224; MR **16**, 609]. The author proves that a one-dimensional homogeneous clan is a (topological) group. *Haskell Cohen* (Baton Rouge, La.).

Schwarz, Štefan. On Hausdorff bicompact semigroups. Czechoslovak Math. J. **5(80)** (1955), 1–23. (Russian. English summary)

This paper is concerned with the structure of compact (Hausdorff)semigroups, in the same spirit as, and with some duplication of results of, papers of Numakura [Math. J. Okayama Univ. **1** (1952), 99–108; MR **14**, 18], Wallace [An. Acad. Brasil. Ci. **25** (1953), 335–336; MR **15**, 854], and Koch [Proc. Amer. Math. Soc. **5** (1954), 828–833; MR **16**, 447]. Let S be a compact semigroup, $a \in S$, $A_n = \{a^n, a^{n+1}, \cdots\}$, and $T_a = \bigcap_{n=1}^{\infty} A_n^-$. Then T_a is a commutative group, and the only idempotent in A_1^- is the unit of T_a. For an idempotent f of S, let K_f be the set of all $a \in S$ such that $f \in \{a, a^2, a^3, \cdots\}^-$. Then S is the union of the (pairwise disjoint) sets K_f. As the author has shown earlier, [Czechoslovak Math. J. 3(78) (1953), 7–21; MR **15**, 850], K_f need not be a semigroup even if S is finite. Also, K_f need not be closed. If S is commutative, then all K_f's are semigroups, and, in fact, K_f is the largest subsemigroup of S containing no idempotent other than f. If all idempotents f of S are primitive ($fx = xf = x$ for an idempotent x implies $x = f$), then all K_f are closed. Let $a \in S$ and e_a be the idempotent in A_1^-. Then S is the union of disjoint groups if and only if $e_a a = a$ for all $a \in S$.

The author also examines carefully the case in which some K_f fail to be closed. For this purpose, he writes $\mathfrak{P}_f = \bigcup\{K_e : e$ is idempotent, $f \in K_f^-\}$ and $\mathfrak{Q}_f = \bigcup\{K_e : e$ is idempotent, $e \in K_f^-\}$. The sets $\mathfrak{P}_f$ and $\mathfrak{Q}_f$ turn out to have a number of useful properties. Sample result: Let S be connected and contain n idempotents, $1 < n < \aleph_0$. Then for all idempotents f, $K_f \neq \mathfrak{P}_f$ or $K_f \neq \mathfrak{Q}_f$. *E. Hewitt.*

Schwarz, Štefan. Characters of bicompact semigroups. Czechoslovak Math. J. **5(80)** (1955), 24–28. (Russian. English summary)

Let S be a compact commutative (Hausdorff) semigroup. Theorem: There is a unique idempotent e in S such that $ef = e$ for all idempotents $f \in S$. Let $\mathfrak{n}$ be the largest subgroup of S containing e [see the preceding review]. The set S^* of all continuous functions χ on S of absolute value 1 such that $\chi(xy) = \chi(x)\chi(y)$ (characters) is clearly a group under multiplication. Theorem: S^* is isomorphic to the character group of the compact Abelian group $\mathfrak{n}$. (In fact, every character of $\mathfrak{n}$ admits a unique extension over S that satisfies the functional equation, and this extension is continuous on S.) *E. Hewitt* (Princeton, N.J.).

Schwarz, Štefan. Topological semigroups with one-sided units. Czechoslovak Math. J. **5(80)** (1955), 153–163. (Russian. English summary)

Throughout this review, let S be a compact (Hausdorff) semigroup. Let L^* (R^*, M^*), when it exists, be the obviously unique left (right, 2-sided) proper ideal of S that contains all left (right, 2-sided) proper ideals of S. Theorem: If L^* exists, then M^* exists and $L^* = M^*$. If R^* also exists, then $L^* = R^*$. Theorem: Suppose that L^* exists. If $S \cap L^{*\prime}$ contains at least 2 elements or S is connected, then $S \cap L^{*\prime}$ is a closed subsemigroup of S containing no proper left ideal. Also $S \cap L^{*\prime}$ is the union of disjoint isomorphic compact groups. This last result is a refinement of a theorem of Faucett, Koch, and Numakura [Duke Math. J. **22** (1955), 655–661, Corollary 1; MR **17**, 282]. Theorem: If L^* exists and $S \cap L^{*\prime}$ contains at least 2 elements, then S contains a right unit. Theorem: Let S be connected. Then L^* exists if and only if S has a right unit and S contains a proper left ideal. Theorem: Suppose that S is connected and that L^* exists. The maximal subgroup containing an arbitrary right unit for S [see the 2nd preceding review] is contained in the boundary of L^*. The author answers in the negative a query of Wallace by constructing a connected compact semigroup with a unique left unit and no right unit. [For another construction of this kind, almost identical to that given here, see the paper reviewed below. In connection with the paper under review see Koch and Wallace, Duke Math. J. **21** (1954), 681–685; MR **16**, 112.] *E. Hewitt.*

Kimura, Naoki; and Tamura, Takayuki. Counter examples to Wallace's problem. Proc. Japan Acad. **31** (1955), 499–500.

The authors give examples to show the existence of a compact connected Hausdorff space which admits a continuous associative multiplication with a unique left unit which is not a right unit. *A. D. Wallace.*

See also: Hlawka, p. 594.

Lie Groups, Lie Algebras

Montgomery, D.; Samelson, H.; and Zippin, L. Singular points of a compact transformation group. Ann. of Math. (2) **63** (1956), 1–9.

Let M be an n-dimensional manifold and G a connected

compact Lie group acting on M as a transformation group. Let s denote the highest dimension of the orbit of any point in M under the group G and Q the set of all points in M which are on orbits of dimension s. It is known that Q is an open set in M. In the present paper, the authors prove that the boundary B of Q in M is at most $(n-2)$-dimensional. More precisely, it is proved that if C is any closed subset of B all of whose points are on orbits of some constant dimension s' $(<s)$, then the dimension of C is at most $(n-1)-(s-s')$. It then follows that B can not separate M and, hence, that Q is connected and M is the closure of Q. Therefore, if, for some integer k, the set of points on orbits of dimension $\leq k$ contains an inner point, then $s\leq k$, i.e. every orbit on M is of dimension $\leq k$. This is a generalization of the theorem of Newman for the case $k=0$.

The main idea of the proof is to construct suitable cycles in M using the homology characterization of dimension and to map those cycles into the decomposition space M^* formed by all orbits on M. In carrying out the proof, it is also proved that M^* is $(n-s)$-dimensional.

K. Iwasawa (Cambridge, Mass.).

Graev, M. I. On a general method of computing traces of infinite-dimensional unitary representations of real simple Lie groups. Dokl. Akad. Nauk SSSR (N.S.) **103** (1955), 357–360. (Russian)

If U is an irreducible unitary representation on a Hilbert space of a simple non-compact Lie group G, it must be infinite-dimensional, if non-trivial, so there is no character function for the representation in the usual sense. It was shown by explicit computations, first in the case of the Lorentz group, later for general classes of simple groups, that there existed a function X on the group having many of the properties of the character, and notably the property that if f is a sufficiently differentiable function on G vanishing outside a compact set, then $\int_G U(a)f(a)$ is a Hilbert-Schmidt operator whose trace exists and equals $\int_G X(a)f(a)da$. Formulas for these traces were given in the case of complex groups by Gel'fand and Naĭmark [Trudy Mat. Inst. Steklov. **36** (1950); MR **13**, 722] based on formulations of the irreducible representations on spaces of all square-integrable functions on certain measure spaces such that the operator $\int_G U(a)f(a)da$ is an integral operator, in terms of whose kernel the trace of the operator is readily expressed. However, the representation spaces for the general real group are typically spaces of holomorphic functions, in formulations such as in Dokl. Akad. Nauk SSSR (N.S.) **100** (1955), 1037–1040 [MR **16**, 795] and it is more difficult to obtain explicit formulas for the trace.

In the present note the author gives such a formula for the real forms of the complex unimodular group. The new method is a generalization of the observation that in the case of the real 2×2 group, in which the representation space may be taken as the square-integrable holomorphic functions on the unit disc in the plane relative to a suitable inner product [see Bargmann, Ann. of Math. (2) **48** (1947), 568–640; MR **9**, 133], the representative for the matrix $\begin{pmatrix}\lambda & 0\\ 0 & \lambda^{-1}\end{pmatrix}$, where $|\lambda|<1$, takes the analytic function $f(z)$ into $f(\lambda^2 z)\lambda^n$, and using the functions $\{z^m; m=0, 1, 2, \cdots\}$ as a basis, the pseudo-character described above may be determined as $\sum_{m=0}^{\infty} \lambda^{n+2m}$.

I. E. Segal.

Leti, Giuseppe. Determinazione dei gruppi aggiunti del gruppo di Galilei e di alcuni suoi sottogruppi. Collect. Math. **7** (1954), 121–140.

If c_{ij}^{h} are the structure constants of a continuous group G, the finite transformations of the adjoint group are given by the linear transformations $\mathbf{e}'=e^{C(u)}\mathbf{e}$, where $C(u)=\sum_j c_{ij}^{h}u_j$, the u_j being canonical parameters of the adjoint group. The author obtains these finite transformations in an explicit form for the restricted and extended Galilean groups, evaluating $e^{C(u)}$ by a formula of L. Fantappiè [C.R. Acad. Sci. Paris **186** (1928), 619–621]. Corresponding results using a more recent formula of Fantappiè [An. Acad. Brasil. Ci. **26** (1954), 25–33; MR **16**, 785] have been obtained by G. Arcidiacono [Rend. Mat. e Appl. (5) **14** (1955), 633–654; MR **17**, 437].

D. E. Rutherford (St. Andrews).

Arcidiacono, Giuseppe. Sui gruppi ortogonali negli spazi a tre, quattro, cinque dimensioni. Portugal. Math. **14** (1955), 63–71.

Orthogonal groups being formally identical with their adjoint groups, their finite transformations might be obtained as in the preceding review. To avoid undue calculation a simplified procedure is adopted for evaluating $e^{C(u)}$ in the cases of the orthogonal groups of 3, 4 and 5 dimensions.

These two papers are motivated by a series of papers by Fantappiè [e.g. Rend. Lincei (8) **12** (1952), 285–290, 553–558; MR **14**, 117, 339] on group structure in physics.

D. E. Rutherford (St. Andrews).

Berger, Marcel. Classification des espaces homogènes symétriques irréductibles. C. R. Acad. Sci. Paris **240** (1955), 2370–2372.

A real homogeneous space $M=G/H$ is defined to be symmetric if H is the set of fixed points of an involutive automorphism $\hat{\sigma}$ of G. This structure is determined locally by the real Lie algebra $\mathfrak{g}$, and an involutive automorphism σ of $\mathfrak{g}$ whose set of fixed points is $\mathfrak{h}$; the latter structure is denoted by $\mathfrak{g}/\mathfrak{h}$. It is defined to be irreducible if, writing $\mathfrak{g}=\mathfrak{h}+\mathfrak{m}$, ad$(\mathfrak{h})$ is irreducible on $\mathfrak{m}$. The classification of irreducible $\mathfrak{g}/\mathfrak{h}$ reduces easily to the case where $\mathfrak{g}$ is simple. Now consider that case. Let $\tilde{\mathfrak{g}}$ be the complexification of $\mathfrak{g}$, τ the conjugation whose set of fixed points is $\mathfrak{g}$, $\mathfrak{g}_u$ a compact real form of $\tilde{\mathfrak{g}}$, $\mathfrak{g}_1=\mathfrak{g}\cap\mathfrak{g}_u$. It is shown that σ is an automorphism of $\mathfrak{g}_1$, thus each $\mathfrak{g}/\mathfrak{h}$ gives rise to an automorphism of $\mathfrak{g}_1$; and distinct $\mathfrak{g}/\mathfrak{h}$ give rise to distinct automorphisms. Using this a complete list is found for all such $\mathfrak{g}/\mathfrak{h}$ where $\mathfrak{g}$ is any non-exceptional simple Lie algebra.

W. Ambrose (Cambridge, Mass.).

Berger, Marcel. Structure et classification des espaces homogènes symétriques à groupe d'isométries semi-simple. C. R. Acad. Sci. Paris **241** (1955), 1696–1698.

We use the notation of the preceding review. It is shown that any such $\mathfrak{g}/\mathfrak{h}$ (with a σ as above) gives rise, for any connected Lie group G whose Lie algebra is $\mathfrak{g}$, to an $M=G/H$ and $\hat{\sigma}$ as above. For given $\mathfrak{g}/\mathfrak{h}$ the set of such M is the set of covering spaces of $M_a=G_a/H_a$ where G_a is the adjoint group of $\mathfrak{g}$. From this it is deduced that any such M can be fibred with a compact symmetric base space and Euclidean fiber. The list of the preceding paper of irreducible structures $\mathfrak{g}/\mathfrak{h}$ is completed by a list of all irreducible $\mathfrak{g}/\mathfrak{h}$ for which $\mathfrak{g}$ is an exceptional simple Lie algebra

W. Ambrose (Cambridge, Mass.).

Stoka, Marius I. Sur les groupes continus finis réels G_{2r} associés aux groupes complexes G_r. Com. Acad. R. P. Romîne 5 (1955), 949–953. (Romanian. Russian and French summaries)

Let G_r be a Lie group of r complex parameters operating in a space of n complex dimensions. There is associated to G_r a Lie group G_{2r} of $2r$ real parameters operating in a space of $2n$ real dimensions. The present note gives expressions for the infinitesimal elements and structure constants of G_{2r} in terms of those of G_r. *P. A. Smith.*

Dixmier, J. Cohomologie des algèbres de Lie nilpotentes. Acta Sci. Math. Szeged 16 (1955), 246–250.

Let L be a nilpotent Lie algebra of finite dimension n over a field F, and let M be an L-module. An L-module is said to be subordinate to M if it is a factor module of a submodule (or, equivalently, a submodule of a factor module) of M. The main results are the following two theorems concerning the cohomology groups $H^i(L, M)$ of L in M. (1) Suppose that F is infinite, and that no non-zero L-module subordinate to M is annihilated by L. Then $H^i(L, M)=(0)$, for all i. (2) Suppose that there exists a non-zero L-module subordinate to M that is annihilated by L. Then $H^i(L, M)\neq(0)$, for all $0<i<n$; and

$$\dim_F (H^i(L, M))\geqq 2, \text{ for all } 0<i<n.$$

It is stated that these results no longer hold for all solvable Lie algebras. *G. Hochschild* (Berkeley, Calif.).

Lazard, M. Sur les algèbres enveloppantes universelles de certaines algèbres de Lie. Publ. Sci. Univ. Alger. Sér. A. 1 (1954), 281–294 (1955).

This paper develops a preliminary note with the same title [C.R. Acad. Sci. Paris 234 (1952), 788–791; MR 13, 719]. Let L be a Lie algebra over an arbitrary commutative ring R. The universal enveloping algebra A of L may be defined as the factor algebra of the tensor algebra built over L, modulo the ideal generated by the elements of the form $x\otimes y-y\otimes x-[x, y]$. The problem is to decide whether or not the canonical map of L into A is a monomorphism. In the case where R is a field this is indeed the case, by the well-known result of Poincaré, Witt, Birkhoff. Here it is shown that this result holds also whenever the R-module L is an inductive limit of a family of cyclic R-modules (i.e., R-modules generated by a single element). This implies that the result holds whenever R is a principal ideal ring. The main feature of the proof is a comparison of A with the symmetric algebra built over L, using the natural filtration of A and the associated graded algebra. *G. P. Hochschild* (Berkeley, Calif.).

See also: Flanders, p. 573; Herstein, p. 577; Massey, p. 653; Švarc, p. 654.

Topological Vector Spaces

Ghika, Al. Propriétés de la convexité dans certains modules. Com. Acad. R. P. Romåne 3 (1953), 355–360. (Romanian. Russian and French summaries)

The author extends to A-modules in which the concept of A-convexity is defined [see Ghika, Acad. R. P. Române. Bul. Şti. Secţ. Şti. Mat. Fiz. 5 (1953), 49–73; Com. Acad. R. P. Române 2 (1952), 329–332; MR 17, 386] two propositions on convex sets which are valid in the ordinary case [see Leĭbenzon, Uspehi Mat. Nauk 7 (1952) no. 2(48), 165–167; MR 13, 848; G. Marinescu, Com. Acad. R. P. Române 3 (1953), 301–303; MR 17, 63]. *G. K. Kalisch.*

Hukuhara, Masuo; et Sibuya, Yasutaka. Sur l'endomorphisme complètement continu. Proc. Japan Acad. 31 (1955), 595–599.

The authors formulate the Riesz theory of completely continuous linear operators, in a linear space which has an ($\mathfrak{L}$)-topology in the sense of Fréchet [Les espaces abstraits, Gauthier-Villars, Paris, 1928], and in which addition and scalar multiplication satisfy suitable continuity assumptions. Such spaces include linear topological spaces, in which the theory was treated by Leray [Acta Sci. Math. Szeged 12, Pars B (1950), 177–186; MR 12, 32] and the reviewer [J. London Math. Soc. 29 (1954), 149–156; MR 15, 801]. [An example of such a space, which is not a linear topological space, is provided by Mikusiński's field of operators; see K. Urbanik, Studia Math. 14 (1954), 243–246; MR 16, 935.] The scope of the present note is approximately the same as that of Leray's paper [loc. cit.]. Proofs of most of the results are indicated; the methods appear to be straightforward.

J. H. Williamson (Princeton, N.J.).

Edwards, David Albert. Les intégrales de Fourier-Stieltjes dans un espace de Banach. C. R. Acad. Sci. Paris 242 (1956), 721–723.

Let X be a complex, separable, weakly complete Banach space with conjugate space X^*. Let $z=z(t)$ be a function with domain $(-\infty, \infty)$ and values in X such that $\|\sum_{r=1}^{n} [z(t_{2r})-z(t_{2r-1})]\|$ is bounded for all sequences $-\infty<t_1<t_2<\cdots<t_{2n}<\infty$. Then the integral $\int_a^b e^{itu}dz(t)$ exists as a strong integral for all $-\infty<a<b<\infty$, and (1) $\int_{-\infty}^{\infty} e^{itu}dz(t)=x(t)$ exists as the strong limit of the finite integral as $a\to-\infty$, $b\to\infty$. Theorem: A function $x(t)$ defined on $(-\infty, \infty)$ with values in X admits a representation (1) if and only if, for every $x^* \in X^*$, there exists a function $\alpha(u; x^*)$ $(-\infty<u<\infty)$ of finite variation on $(-\infty, \infty)$ such that $x^*(x(t))=\int_{-\infty}^{\infty} e^{itu}d\alpha(u; x^*)$. A corollary of this theorem is a variant of the representation theorem of Phillips for Banach-space-valued Fourier-Stieltjes transforms [Trans. Amer. Math. Soc. 69 (1950), 312–323; MR 12, 496]. No proofs are given. *E. Hewitt.*

See also: Wolfson, p. 647.

Banach Spaces, Banach Algebras

Fréchet, Maurice. Conditions d'existence (exprimées en termes de „variations généralisées") d'un extremum local d'une fonctionnelle. C. R. Acad. Sci. Paris 241 (1955), 1901–1904.

This note contains a statement of elementary conditions for an extreme of a function defined on a Banach space and having generalized nth-order variations.

L. M. Graves (Chicago, Ill.).

Ionescu Tulcea, C. T. Deux théorèmes concernant certains espaces de champs de vecteurs. Bull. Sci. Math. (2) 79 (1955), 106–111.

Let Z be a locally compact space, μ a positive Radon measure on Z and $\mathcal{E}=\{E(z)|z \in Z\}$ a family of Banach spaces. Let $\mathcal{C}(\mathcal{E})$ be the linear space of all functions x defined on Z with $x(z) \in E(z)$ for each $z \in Z$. Following Godement [Ann. of Math. (2) 53 (1951), 68–124; MR 12, 421] (who took each $E(z)$ to be a Hilbert space) the author considers a "fundamental" set $\mathcal{A}\subseteq\mathcal{C}(\mathcal{E})$ and an associated Banach space $L_{\mathcal{A}}^p$ $(1\leqq p<\infty)$. The space $L_{\mathcal{A}}^p$ consists of elements of $\mathcal{C}(\mathcal{E})$ normed by $\|x\|_p=(\int\|x(z)\|^p d\mu(z))^{1/p}$.

In the presence of a certain denumerability hypothesis, the author shows that every linear continuous functional f on $L_{\mathcal{J}}^p$ has the form $f(x)=\int\langle x, x_f'\rangle d\mu$ ($x \in L_{\mathcal{J}}^p$) where $x_f' \in \mathcal{C}(\mathcal{E}')$, $\mathcal{E}'=\{E'(z)|z \in Z\}$, $E'(z)$ being the dual of $E(z)$, and $\|f\|=\|x_f'\|_{p'}$ $(1/p+1/p'=1)$. This result (without denumerability hypothesis) is due to Godement [loc. cit.] for the case of Hilbert spaces. The author's proof depends on a similar extension of a theorem on decomposition of operators which was proved by von Neumann [ibid. **50** (1949), 401–485; MR **10**, 548] and Godement [loc. cit.].

C. E. Rickart (New Haven, Conn.).

Deny, J.; et Lions, J. L. Les espaces du type de Beppo Levi. Ann. Inst. Fourier, Grenoble **5** (1953–1954), 305–370 (1955).

The Beppo Levi type spaces consist of functions defined in a domain whose partial derivatives in the weak sense of some order (usually first) belong to some class, mainly L_2 or L_p. The first part of this paper is concerned with inequalities between L_2 and L_p norms of functions, their derivatives and their boundary values; these inequalities are related to those of Sobeleff and Poincaré. The orthogonal decomposition with respect to the space of harmonic functions is used as a tool. The second part contains a refined real variable investigation of B. L. functions, in particular their behavior near the boundary. The results were announced in C.R. Acad. Sci. Paris **239** (1954), 1174–1177 [MR **16**, 718]. *P. D. Lax* (New York, N.Y.).

Rooney, P. G. A generalization of some theorems of Hardy. Trans. Roy. Soc. Canada. Sect. III. (3) **49** (1955), 59–66.

Let $\varphi \in L_p(0, \infty)$ and let

$$\xi(x)=\frac{1}{x}\int_0^x \varphi(t)dt,\quad \eta(x)=\int_x^\infty \frac{\varphi(t)}{t}dt.$$

It was proved by Hardy [Messenger of Math. **54** (1925), 150–156; **57** (1927), 12–16] that $\xi \in L_p(0, \infty)$ for $1<p\leqq\infty$ and $\eta \in L_p(0, \infty)$ for $1\leqq p<\infty$. In this paper similar theorems are proved for φ a member of $\Lambda(\alpha, p)$ or $M(\alpha, p)$ where $\Lambda(\alpha, p)$, $M(\alpha, p)$ are the spaces defined by G. G. Lorentz [Ann. of Math. (2) **51** (1950), 37–55; MR **11**, 442]. Applications of these theorems are given to Laplace transforms of functions in $\Lambda(\alpha, p)$ and $M(\alpha, p)$. *R. E. Fullerton.*

Ellis, H. W.; and Halperin, Israel. Haar functions and the basis problem for Banach spaces. J. London Math. Soc. **31** (1956), 28–39.

The paper consists of two parts: (I) The construction of a system of functions whose properties parallel those of the classical functions of Haar as he defined them for [0, 1]. The development here is analogous to but broader and more extensive than that given by the reviewer in his thesis [Princeton, 1948]. The technique employs a sequence of finite partitions monotonely ordered by refinement. The Haar functions associated with the partition are those which arise essentially from a Gram-Schmidt orthonormalization applied to the characteristic functions of the partition sets. Convergence theorems for "Haar-Fourier" series are obtained. In particular, the generalization of Haar's theorem on the uniform convergence of the expansion of a "continuous" function to the given function. Since no topology is assumed, continuity is phrased in terms of small oscillation on sets of sufficiently refined partitions.

(II) The functions considered in (I) are shown to be a basis for the Banach spaces L^λ, where λ is a "mesh levelling" function of a Haar system. The properties assumed for λ are: (a) $0\leqq\lambda(u)\leqq\infty$ for any measurable non-negative function u; (b) $u=0$ a.e. implies $\lambda(u)=0$; (c) $u_1\leqq u_2$ implies $\lambda(u_1)\leqq\lambda(u_2)$; (d) $\lambda(u_1+u_2)\leqq\lambda(u_1)+\lambda(u_2)$; (e) $\lambda(ku)=k\lambda(u)$ if $k>0$; (f) λ (sup u_n)$=$sup $\lambda(u_n)$ if $u_n\leqq u_{n+1}$. Then L^λ is the set of measurable functions f for which $\|f\|=\lambda(|f(p)|)<\infty$. There are technical details which are required to make the theorems and proofs run smoothly. However, one can say, modulo essentially obvious restrictions, that a Haar-type basis can be constructed for "every" Banach space. This again constitutes a far reaching extension of the reviewer's results in his 1948 thesis.

B. Gelbaum (Minneapolis, Minn.).

Rothe, Erich H. Mapping degree in Banach spaces and spectral theory. Math. Z. **63** (1955), 195–218.

The usual definitions of the topological degree of a linear homeomorphism of a finite-dimensional space yield ±1, depending on the sign of the determinant. For a complex Banach space admitting a conjugation and a real completely continuous (c.c.) linear transformation L, the topological-degree definitions consistent with the above lead to the theorem that if the spectrum, $\sigma(T)$, attached to $T=\lambda I-L$ does not contain $\lambda=1$, and the point spectrum $\{\lambda_i\}$ is finite and of finite multiplicity $\{n_i\}$ then the topological degree is exp $i\pi\sum_1^N n_i$. The author takes this theorem as a definition of the degree and proposes to develop the consequences of such a definition. Thus not only the c.c. case, but also an operator with $\lambda=1$ not in $\sigma(T)$, which splits into one with finite-dimensional ranges, and one whose spectrum lies in Re$(\lambda)<1$ can enter. Really the author introduces a slightly more general category, referred to below as C. The usual homotopy invariance theorem now appears as the assertion that under suitable perturbations, always remaining in C, $\sum_{\mathrm{Re}(\lambda_i)>1}m_i$, doesn't change. It is shown the degree can be extended to a class of smooth non-linear operators — smooth in the sense that $f(x)=x-F(x)$ admits (a) only a finite number of solutions, $\{x_j\}$, in the connected open set D for each y, and (b) bounded Fréchet differentials $(df)(x_j, \delta x)$, so that the degree at y with respect to D is the sum of the degrees for the linear transformations $(df)(x_j, \delta x)$. *D. G. Bourgin* (Rome).

Laugwitz, Detlef. Über einen Abbildungssatz von B. Sz.-Nagy. Math. Z. **64** (1955), 72–78 (1956).

A mapping f of a normed linear space E into itself is called "dehnungsbeschränkt" provided there is a finite M such that $\|fx-fx'\|\leqq M\|x-x'\|$ for all $x, x' \in E$. The author considers an f such that $fx=(\lambda x)x$ for each $x \in E$, λ being a real-valued function having $\lambda x>0$ for $x\neq0$ and $\lambda(rx)=\lambda x$ for $r>0$. He shows that f and f^{-1} are both "dehnungsbeschränkt" if and only if f can be generated in a certain simple way by two sets which are starlike from the origin and subject to certain other restrictions. His theorem extends earlier results of Sz.-Nagy, Vincze, and Szüsz [Vincze and Szüsz, Acta Sci. Math. Szeged **14** (1951), 96–100; MR **12**, 573], Haupt [S.-B. Math.-Nat. Kl. Bayer. Akad. Wiss. **1951**, 147–161; MR **14**, 305], and the reviewer [Amer. Math. Monthly **60** (1953), 618–619; MR **15**, 535].

V. L. Klee, Jr. (Los Angeles, Calif.).

Tamme, È. È. On approximate solution of functional equations by the method of expansion in a series of an inverse operator. Dokl. Akad. Nauk SSSR (N.S.) **103** (1955), 769–772. (Russian)

Let P be an analytic function on a Banach space X to

another such space Y. This paper deals with the possibility of obtaining a solution x^* of $P(x)=0$ by the expansion

$$x^*=x_0+\sum_{k=0}^{\infty}\frac{(-1)^k}{k!}\Phi^{(k)}(y_0)[P(x_0)]^k.$$

Here x_0 is an approximation sufficiently close to x^*, $y_0=P(x_0)$, and Φ is the inverse of P and is assumed to exist near y_0. Bounds on the derivatives $\Phi^{(k)}$ (which can be expressed in terms of the derivatives $P^{(k)}$) and conditions sufficient to assure convergence are given. The error in taking a finite number of terms is estimated. Faster rates of convergence were obtained by L. V. Kantorovič [same Dokl. **59** (1948), 1237–1240; MR **9**, 537] who used the first two terms as an iterative procedure (Newton's method), and by M. I. Nečepurenko [Uspehi Mat. Nauk (N.S.) **9** (1954), no. 2(60), 163–170; MR **15**, 801] using the first three terms iteratively (Čebyšev's method). However the present procedure has the advantage that the only inversion required is $[P'(x_0)]^{-1}$. *R. G. Bartle.*

Kaazik, Yu. Ya.; and Tamme, È. È. On a method of approximate solution of functional equations. Dokl. Akad. Nauk SSSR (N.S.) **101** (1955), 981–984. (Russian)

Let P be an analytic operator mapping a Banach space X onto a linear normed space Y. One wants to find the solution x^* of the equation $P(x)=0$, starting from a given x_0. Let $\Delta x_i=x_i-x_0$. The authors consider the following linearly convergent iterative algorithm determining x_n $(n=1, 2, \cdots)$:

$$P(x_0)+P'(x_0)\Delta x_1=0,$$

$$P(x_0)+[P'(x_0)+\sum_{k=2}^{n}(k!)^{-1}P^{(k)}(x_0)\Delta x_{n-k+1}\cdots\Delta x_{n-1}]\Delta x_n=0.$$

When X, Y are the complex field, the algorithm is known [E. Schröder, Math. Ann. **2** (1870), 317–365, and others]. If $P'(x_0)$ has an inverse Γ with $\|\Gamma\|\leq B$, and if certain other conditions hold, it is proved that $\|x^*-x_n\|\leq Cm^n$, where C and m are explicit constants involving B, $\|\Gamma P(x_0)\|$ and the asymptotic nature of $\|(k!)^{-1}P^{(k)}(x_0)\|$. An additional hypothesis implies the uniqueness of the solution in a neighborhood of x_0.

Two numerical examples deal with the solution of polynomial equations in one and two complex variables, and compare the authors' bounds with the true errors. *G. E. Forsythe* (Los Angeles, Calif.).

Halilov, Z. I. Linear singular equations in a unitary ring. Akad. Nauk Azerbaĭdžan. SSR. Trudy Inst. Fiz. Mat. **3** (1948), 3–28. (Russian. Azerbaijani summary)

The content of this paper is essentially the same as one in Mat. Sb. N.S. **25(67)** (1949), 169–188; MR **11**, 373; **12**, 1002.

Audin, Maurice. Extension de la méthode d'extremum de Galois-Hilbert à des cas non symétriques. C. R. Acad. Sci. Paris **241** (1955), 1197–1198.

For a bounded linear operator A on a Banach space E it is well-known that the modulus of the smallest singularity of $(I-\lambda A)^{-1}$ equals the radius of convergence of $\sum\lambda^n A^n$, considered over E; by considering it over subspaces the moduli of other singularities may, under conditions, be obtained. The author applies this to the case of singularities less in modulus than the Fredholm-radius R_A [cf. J. Radon, Akad. Wiss. Wien. S.-B. IIa. **128** (1919), 1083–1121; Audin, C.R. Acad. Sci. Paris **237** (1953), 511–512; **238** (1954), 2221–2222; MR **15**, 233; **16**, 142]. Also recalled is that in Hilbert space the compactness of A is necessary and sufficient for $R_A=\infty$. *F. V. Atkinson.*

Gohberg, I. C. Some properties of normally soluble operators. Dokl. Akad. Nauk SSSR (N.S.) **104** (1955), 9–11. (Russian)

Let A be a normally soluble linear operator on a Banach space E into E, let $L(A)$ denote the set of $x\in E$ with $Ax=0$, $\alpha(A)$ the dimensionality of $L(A)$, and I the identity operator. It is a question of what additional conditions ensure that $A-\lambda I$ is normally soluble and that $\alpha(A-\lambda I)=\text{const}=m<\infty$ in a complex λ-region of the form $0<|\lambda|<\varrho$. Sets of conditions including the requirement $\alpha(A)<\infty$ have been given by previous writers, including the author [same Dokl. (N.S.) **101** (1955), 9–12, q.v. for further references; MR **17**, 284]. Now given, restricting E to be a Hilbert space and A to be bounded, are the following weakened conditions: $A, A^2, \cdots, A^{k+1}$ are to be normally soluble, and $L(A^{n+1})\ominus L(A^n)$ is to be of dimension m for $n\geq k$. If the latter condition is strengthened to $L(A^{k+1})=L(A^k)$, then $A-\lambda I$ has a bounded left inverse for $0<|\lambda|<\varrho$. In the reviewer's opinion the restriction to Hilbert space is avoidable. *F. V. Atkinson* (Canberra).

Wolfson, Kenneth G. Anti-isomorphisms of the ring and lattice of a normed linear space. Portugal. Math. **14** (1955), 1–7.

Let X_i be a normed linear space, $E(X_i)$ the ring of all bounded linear operators on X_i and $L(X_i)$ the lattice of closed subspaces of X_i $(i=1, 2)$. It is shown that $E(X_1)$ is anti-isomorphic to $E(X_2)$ if and only if $L(X_1)$ is anti-isomorphic to $L(X_2)$ and also if and only if X_1 is pseudo-reflexive and X_2 is its pseudo-conjugate. In case at least one of the X_i is complete then each X_i is reflexive and is the conjugate space of the other. If, in addition, the X_i are infinite-dimensional then the ring (lattice) isomorphism can be identified as the mapping of an operator (subspace) upon its adjoint (annihilator). *B. Yood.*

Wolfson, Kenneth G. Annihilator rings. J. London Math. Soc. **31** (1956), 94–104.

An annihilator ring is a topological ring in which every closed proper left (right) ideal has a non-zero right (left) annihilator. Every dual ring in the sense of Kaplansky [Ann. of Math. (2) **49** (1948), 689–701; MR **10**, 7] is an annihilator ring. In fact, the concept of annihilator ring was introduced (for the special case of Banach algebras) by Bonsall and Goldie [Proc. London Math. Soc. (3) **4** (1954), 154–167; MR **15**, 881] as a generalization of dual rings. If A, B are topological vector spaces over the same division ring F, and if there is a continuous bilinear function on $A\times B$ to F with respect to which A and B are dual, then A and B are called dual topological vector spaces.

The main theorems of this paper are: (1) If K is a simple annihilator ring with a closed maximal right ideal, then there exist a pair A, B of dual topological vector spaces such that K contains as a dense (topologically) subring the ring of all linear transformations of finite rank on A which have adjoints in B. (2) If K is an annihilator ring such that the intersection of its closed regular maximal ideals is zero, then K contains as a dense subring the direct sum of its minimal closed two-sided ideals. In addition, a partial "converse" of (1) above is obtained, some of the

results of Bonsall and Goldie are proved more simply, and the structure of discrete annihilator rings is described.

M. Henriksen (Lafayette, Ind.).

Herz, Carl S. Spectral synthesis for the Cantor set. Proc. Nat. Acad. Sci. U.S.A. **42** (1956), 42–43.

For $f \in L_1(-\infty, \infty)$ and y a real number, let $\hat{f}(y)=\int_{-\infty}^{\infty} e^{-ixy}f(x)dx$. Let I be a closed ideal in the group algebra $L_1(-\infty, \infty)$ such that $\{y: \hat{f}(y)=0 \text{ for all } f \in I\}$ is Cantor's ternary set, C. Then $I=\{f: f \in L_1(-\infty, \infty), \hat{f}(y)=0 \text{ for } y \in C\}$. This remarkable result is equivalent to spectral synthesis for the Cantor set, and is the first result of its kind known to the reviewer. [For background and other results, see H. Pollard, Duke Math. J. **20** (1953), 499–512; MR **15**, 215.] *E. Hewitt* (Princeton, N.J.).

Takeda, Zirô. Inductive limit and infinite direct product of operator algebras. Tôhoku Math. J. (2) **7** (1955), 67–86.

The author begins by introducing an inductive limit of C^*-algebras which is used to extend the work of Turumaru [same J. (2) **4** (1952), 242–251; **5** (1953), 1–7; **6** (1954), 208–211; MR **14**, 991; **15**, 237; **16**, 1126] and Misonou [ibid. **6** (1954), 189–204; MR **16**, 1125] to infinite products. Although it is shown that the product is independent of the underlying Hilbert spaces, this property is lost if one tries to extend to a w^*-limit.

To handle the w^*-case, a w^*-algebra is the w^*-inductive limit of a directed set of w^*-algebras provided all "normal" states (σ-weakly continuous states) of the limit are the projective limits of the "normal" states of the approximating algebras. Unfortunately the author is unable to prove that the direct product of two directed sets of w^*-approximating algebras converges to the product of their limits, although the limits are algebraically unique.

Finally this w^*-inductive limit is used to define an infinite product of w^*-algebras, of which the Murray-von Neumann product is a homomorphic image [Ann. of Math. (2) **37** (1936), 116–229]. Although the author's product is independent of the spaces, he has not been able to prove its associativity. As application of the above theory, he obtains other results of which this is typical: The spectrum X of the infinite direct product of commutative C^*-algebras with identities is homeomorphic to the product of the spectra of the component algebras. [For similar results see Segal, Amer. J. Math. **76** (1954), 721–732; MR **16**, 149; Shields, Bull. Amer. Math. Soc. **60** (1954), 52.] *E. L. Griffin, Jr.* (Ann Arbor, Mich.).

Tsuji, Kazô. Representation theorems of operator algebra and their applications. Proc. Japan Acad. **31** (1955), 272–277.

The author announces (proofs will be published later) some extensions of the work of M. Tomita [Math J. Okayama Univ. **3** (1954), 147–173, in particular theorems 5, p. 161 and 6, p. 163; MR **15**, 968]. [See also Godement, J. Math. Pures Appl. (9) **30** (1951), 1–110; Ann. of Math. (2) **59** (1954), 63–85; MR **13**, 12; **15**, 441; and H. Umegaki, Jap. J. Math. **22** (1952), 27–50; MR **16**, 49.] The theorems concern expansions of states of C^*-algebras into integrals of states on component C^*-algebras, and the extension consists of the removal of the condition that the algebras have identities. As application the following is stated in notation of Tomita. Theorem 3. Let G be a locally compact group, let $N(G)$ be the set of all elementary positive definite functions X on G with $\|X\|=1$ and let $\overline{N(G)}$ denote the weak closure of $N(G)$. Given a continuous positive definite function ϕ on G and the diagonal algebra E on $L^2(\phi)$. Then there exists a Borel measure π with the following properties: (1) $\overline{N(G)}-N(G)$ has π-measure 0. (2) Let $M(G)$ be $*$-algebra of all essentially bounded π-measurable functions ξ on $\overline{N(G)}$; then, for every $\xi \in M(G)$ there exists $K_\xi \in E$ which satisfies $(U_{a:\phi}K_\xi\hat{\phi}, \hat{\phi})=\xi(x)X(a)d\pi(X)$ for every $a \in G$, $U_{a:\phi}$ the representation of $L_1(G)$ on $L_2(\phi)$. (3) The correspondence $\xi \rightarrow K_\xi$ is a $*$-algebraic isomorphism of $M(G)$ onto E. (4) If ϕ is a continuous central positive definite function on G, then almost all X in reference to the measure π are elementary continuous central positive definite functions.

E. L. Griffin, Jr. (Ann Arbor, Mich.).

See also: Tsuji, p. 603; Alexiewicz and Orlicz, p. 611; Koševel, p. 620; Fichera, p. 626.

Hilbert Space

Nevanlinna, F. [R.]; und Nieminen, T. Das Poisson-Stieltjes'sche Integral und seine Anwendung in der Spektraltheorie des Hilbert'schen Raumes. Ann. Acad. Sci. Fenn. Ser. A. I. no. **207** (1955), 38 pp.

The authors give an alternate simplified "moment-theoretic" proof of the spectral resolution theorem for unitary and for self-adjoint operators in Hilbert space. As a fundamental moment theorem they use the Poisson-Stieltjes representation of a positive harmonic function in the unit circle (unitary case) and in a half-plane (self-adjoint case) for which they supply simplified proofs. These representation formulae are then applied to appropriate quadratic forms defined in terms of the resolvent of a unitary or a self-adjoint operator, to give representation formulae from which the spectral theorem can be derived by standard arguments. *J. Schwartz.*

Slobodyanskiĭ, M. G. On the construction of an approximate solution in linear problems. Prikl. Mat. Meh. **19** (1955), 571–588. (Russian)

In many linear problems, it is required to determine approximately the scalar product $(f, v)=(u, \psi)$, where u and v are solutions of the equations $Au=f$, $A^*v=\psi$, with A a linear operator defined on a Hilbert space H, and A^* the adjoint operator of A, and f and ψ given elements of the Hilbert space H. This point of view has been developed by the author in earlier papers [Prikl. Mat. Meh. **15** (1951), 245–250; **16** (1952), 449–464; MR **13**, 288; **14**, 502] with special emphasis on self-adjoint problems. In the present paper, after obtaining certain bounds in the not necessarily self-adjoint case, the author applies his results to the following boundary-value problem for an ordinary differential equation:

$$Au=\sum_{i=0}^{\nu} A_i u=f(x) \left(A_i u=(-1)^i \frac{d^i}{dx^i}\left(p_i \frac{d^i u}{dx^i}\right)\right),$$

$$u(a)=\cdots=u^{(\nu-1)}(a)=0, \quad u(b)=\cdots=u^{(\nu-1)}(b)=0,$$

with $p_\nu>0$, $p_i \geqq 0$ on $a \leqq x \leqq b$; and to the determination of the deflection of a clamped elastic plate. Several numerical examples are given. *J. B. Diaz* (College Park, Md.).

See also: Cimmino, p. 604; Graev, p. 644; Ionescu Tulcea, p. 645; Audin, p. 647; Gohberg, p. 647; Visconti, p. 693.

TOPOLOGY

Have

★ **Hall, Dick Wick; and Spencer, Guilford L., II. Elementary topology.** John Wiley & Sons, Inc., New York; Chapman & Hall, Ltd., London. 1955, xii+303 pp. $7.00.

The "advanced undergraduates" for whom this textbook is written, are fortunate to have such an aid. There are seven chapters, entitled as follows: I. Introductory set theory. II. The real line. III. Topological spaces. IV. Metric spaces. V. Arcs and curves. VI. Partitionable spaces. VII. The Axiom of Choice. Chapters I and VII consist of "naïve" set theory, covering what the nonlogician needs concerning Zorn's Lemma, the Axiom of Choice and the Tychonoff product theorem, with some applications; but transfinite arithmetic is not mentioned. In Chap. II, the reader is given the standard topics (e.g. Heine-Borel theorem, upper and lower bounds) needed by students starting rigourous analysis. Topologists who have to teach Analysis à la Hardy's "Pure mathematics" will find the treatment here more to their taste. The material also provides a good introduction to Chap. III which, with an axiomatic approach leading to Cartesian products and compactness, gives way naturally to metric spaces in Chap. IV. Here, the level of "general topology" reached may be judged by the fact that two metrisation theorems are proved (Urysohn's and "a space is metric if and only if it has a K-basis") although no technical properties of coverings like local finiteness are introduced. At this point, the reviewer felt that the elementary and attractive parts of the Lebesgue-covering dimension theory might have been included.

Chap. V is more geometric than the rest of the book, and gives the classical theorems concerning characterisation of arc, Jordan curve and 2-sphere, all proved by non-algebraic methods. The proofs are given in great detail with a reasonable amount of motivation, and this chapter is the longest in the book (over 100 pages). In it, too, the reader meets for the first time local compactness and one-point compactification. Chap. VI gives the elementary theory of partitioning of spaces, in particular apparently, to prepare the reader for the study of Bing's recent work on the characterisation of the 3-sphere. A statement and skeleton proof of this result would have been preferable here, as the material of the chapter as given is otherwise only of mainly technical interest.

The book contains 430 problems scattered through the chapters. These vary from the trivial to, for example, the suggestion that the student write a "term paper" on Moore-Smith convergence, following the skeleton outline given by the authors. These and the many counterexamples given in the text, as well as the patient way (sometimes repetitive but not tedious) in which the reader is led on, are very helpful features indeed.

The reviewer has some minor criticisms to make. In Chapters II and III, continuity is defined in ways which are equivalent but so different that the student might be confused without some explanation. In Chap. II as well, the real line is assumed implicitly to exist, although tucked away in Chap. IV it is pointed out that the reals may be constructed as the completion of the rationals; here again a few words concerning the continuity of the algebraic operations might well have been included. (Each chapter ends with a relevant bibliography, but only from the works there cited could the student learn of even the existence of topological algebra or algebraic topology.) Also, why use the mystifying term "second countable space" without explaining the historical reason for the abbreviation? These examples are really included in the general remark that the authors are not good on salesmanship. A reader with no informed contacts might well wonder on the evidence of the book what motivates topologists at all, and why topology should be a major branch of mathematics (as the authors mention in the preface). But with the book and a contact strong on proppaganda, the student is well catered for, whether he intends to be a topologist — "point-set" or "algebraic" — or not. *H. B. Griffiths* (Bristol).

★ **Monteiro, António A. L'arithmétique des filtres et les espaces topologiques.** Segundo symposium sobre algunos problemas matemáticos que se están estudiando en Latino América, Julio, 1954, pp. 129–162. Centro de Cooperación Cientifica de la UNESCO para América Latina, Montevideo, Uruguay, 1954.

On order

Un réticulé est, par définition, un ensemble R partiellement ordonné (par une relation notée $\leqq$) et dont chaque couple d'éléments possède une borne inférieure et une borne supérieure appartenant à R. L'A. appelle filtre d'un réticulé R toute famille F non vide d'éléments de R telle que tout élément de R qui est $\geqq$ un élément de F appartienne à F, et que la borne inférieure de deux éléments quelconques de F appartienne à F. L'ensemble $\Phi=\Phi(R)$ des filtres d'un réticulé R sera considéré comme partiellement ordonné en convenant que deux filtres F_1 et F_2 de R vérifient $F_1\leqq F_2$ si et seulement si $F_1\supset F_2$. Alors Φ est un réticulé. Deux ensembles partiellement ordonnés sont dits isomorphes, ou ayant le même type d'ordre partiel, s'il existe une transformation biunivoque T de l'un en l'autre telle que T et la transformation réciproque de T conservent toutes deux la relation d'ordre partiel.

L'A. fait correspondre à tout espace accessible E de M. Fréchet le réticulé $R=R(E)$ de tous les ensembles fermés de E, R étant partiellement ordonné par la relation d'inclusion $\subset$. Il appelle filtres fermés de E les filtres du réticulé $R=R(E)$. Il appelle réticulé des filtres fermés de E le réticulé $\Phi=\Phi(R)=\Phi(R(E))$ dont les éléments sont les filtres fermés de E. Ceci étant, l'A. cherche à caractériser diverses classes d'espaces accessibles E par des propriétés du type d'ordre partiel de $R(E)$, ou de préférence de $\Phi(R(E))$.

Il résout ce problème pour les espaces normaux, les espaces accessibles extrêmement disconnexes (c'est-à-dire les espaces accessibles vérifiant l'axiome de Stone à savoir: toute fermeture d'un ensemble ouvert est un ensemble ouvert), et les espaces complètement normaux. Il laisse ce problème irrésolu pour les espaces complètement réguliers. L'A. donne des conditions nécessaires et suffisantes pour qu'un réticulé soit isomorphe au réticulé des filtres fermés d'un espace de Hausdorff parfaitement compact (au sens de M. Fréchet), respectivement d'un espace distanciable compact.

Les propriétés du réticulé des filtres fermés d'un espace accessible E deviennent d'autant plus analogues à celles du réticulé des entiers naturels (ce dernier étant ici partiellement ordonné par la convention qu'un entier a est $\leqq$ un entier b si et seulement si a divise b au sens de l'arithmétique) que E vérifie des axiomes topologiques de séparation plus stricts. Cette analogie est déjà grande quand E est complètement normal; elle est maximum et presque complète quand E est un espace hypernormal (on appelle espace hypernormal un espace à la fois complètement

normal et extrêmement disconnexe). Ces faits montrent le caractère arithmétique des axiomes topologiques de séparation, et permettent de dire que les filtres fermés jouent dans la théorie des espaces topologiques un rôle ayant quelque analogie avec celui des entiers naturels dans la théorie du continu lineaire.

L'A. traite aussi pour eux-mêmes bien des problèmes se rapportant aux réticulés; et les correspondance $E \to R(E)$ et $E \to \Phi(R(E))$ l'amènent à transposer dans la théorie des réticulés bien des notions concernant les espaces topologiques, et réciproquement. Il applique ses résultats à des problèmes topologiques de compactification, et à des questions de logique mathématique. *A. Appert.*

Ishikawa, Fumie. On countably paracompact spaces. Proc. Japan Acad. 31 (1955), 686–687.

The author obtains the following characterization of countably paracompact spaces which, if the space is normal, reduces to a well-known characterization given by C. H. Dowker [Canad. J. Math. 3 (1951), 219–224, Th. 2(d); MR 13, 264]: A topological space is countably paracompact if and only if, for every decreasing sequence F_i of closed subsets with empty intersection, there exists a decreasing sequence G_i of open sets whose closures have empty intersection, such that $G_i \supset F_i$ for all i.
E. Michael (Seattle, Wash.).

Kuratowski, K. Sur l'espace des fonctions partielles. Ann. Mat. Pura Appl. (4) 40 (1955), 61–67.

Let X, Y be two metric spaces of which X is compact. Let $P(X, Y)$ denote the set of all continuous functions f each of which is defined on some non-empty closed subset $A(f)$ of X and takes its values in Y. Let Γ_f designate the graph of the function f: $\Gamma_f = \{(x, y) \in A(f) \times Y : y = f(x)\}$. The author takes the Hausdorff distance of the two bounded closed subsets Γ_f, Γ_g of the product metric space $X \times Y$ as the distance of f, g in $P(X, Y)$. It is shown that in this metric space $P(X, Y)$, a sequence $\{f_n\}$ of elements converges to f, if and only if the following two conditions are satisfied: (1) the sequence $\{A(f_n)\}$ of sets converges to the set $A(f)$ in the sense of the Hausdorff distance for bounded closed subsets of X; (2) $\lim x_n = x$, $x_n \in A(f_n)$ and $x \in A(f)$ imply $\lim f_n(x_n) = f(x)$ (i.e. continuous convergence). As example, the author discusses a type of convergence for sequences of locally connected continua studied previously by S. Mazurkiewicz [Fund. Math. 24 (1935), 118–134]. *Ky Fan* (Notre Dame, Ind.).

Nagata, Jun-iti. On coverings and continuous functions of topological spaces. Proc. Japan Acad. 31 (1955), 688–693.

The author has characterized metric spaces by means of a family $\{f_\alpha\}_{\alpha \in A}$ of continuous, real-valued functions on the space which formed a partition of unity (i.e. $0 \leqq f_\alpha \leqq 1$, and $\sum_\alpha f_\alpha = 1$) [J. Inst. Polytech. Osaka City Univ. Ser. A. 1 (1950), 93–100; MR 13, 264], and the reviewer has obtained an analogous characterization for paracompact spaces [Proc. Amer. Math. Soc. 4 (1953), 831–838; MR 15, 144]. The first part of the present paper is devoted to similar characterizations, but instead of assuming that $\{f_\alpha\}_{\alpha \in A}$ is a partition of unity, it is assumed that $\sup_{\alpha \in B} f_\alpha$ (and, in the metric case, $\inf_{\alpha \in B} f_\alpha$) is continuous for all $B \subset A$.

In the second part of the paper, the author proves the following theorem which, for metric spaces, is due to Hausdorff [Fund. Math. 30 (1938), 40–47]: Let R and S be uniform spaces, both having a base for the uniformity of the same cardinality $\mathfrak{m}$, and let R be paracompact. If now $A \subset R$ is closed, and $f: A \to S$ continuous, then S can be embedded as a closed sub-uniform space in a uniform space T having a base for the uniformity of cardinality $\mathfrak{m}$, such that f has a continuous extension $g: R \to T$ which maps $R - A$ homeomorphically onto $T - S$; if f is a homeomorphism, then so is g. *E. Michael* (Seattle, Wash.).

Motchane, L. Propriétés invariantes par convergence simple. J. Math. Pures Appl. (9) 34 (1955), 337–394.

Ce mémoire comprend trois chapitres. Les principaux résultats des chapitres I et III ont été, en substance, annoncés aux C.R. Acad. Sci. Paris 231 (1950), 1206–1208; 233 (1951), 1569–1571 [MR 13, 21, 857]. Le chapitre II est consacré à l'étude des applications d'un produit fini d'espaces métriques complets séparables X_i dans un espace métrique séparable Z qui sont séparément (partiellement) continues. Ces applications sont quasi-continues. Le théorème de Baire relatif à l'existence d'un réseau de droites de continuité quand $n=2$, et que X_1, X_2 et Z sont la droite numérique, leur est étendu. Toutes les démonstrations sont données. *C. Y. Pauc* (Nantes).

★ **Gottschalk, Walter Helbig; and Hedlund, Gustav Arnold. Topological dynamics.** American Mathematical Society Colloquium Publications, Vol. 36. American Mathematical Society, Providence, R. I., 1955. vii+151 pp. \$5.10.

"By topological dynamics we mean the study of transformation groups with respect to those topological properties whose prototype occurred in classical dynamics" (authors' preface). A topological transformation group is defined to be a triple consisting of a topological space X, a topological group T and a continuous mapping $(x, t) \to xt$ from $X \times T$ into X such that $(xt)s = x(ts)$ and $xe = x$, where e is the identity in T.

In part I, which consists of 11 sections, various concepts and theorems of dynamical systems are generalized to suitably extensive classes of topological transformation groups. Isomorphisms, homomorphisms, orbits, orbit closures, invariant subsets, etc., of topological transformation groups are defined, as are partitions and decompositions of spaces. Theorems are given stating when the orbit closures constitute a partition of X.

A subset A of T is called left syndetic in T if $T = AK$ for some compact subset K of T. The period of T at x is defined to be the greatest subset P of T such that $xP = x$. The group T is said to be periodic at x if P is a syndetic subset of T. The group T is said to be recursive at x if for each neighbourhood U of x there is an 'admissible' subset A of T such that $xA \subset U$. Various generalizations of periodicity such as almost periodicity, regular almost periodicity and recurrence are obtained as special cases of recursion by choosing suitable classes of admissible sets. Thus many theorems can be stated and proved for all these kinds of recursions at once.

In sections 4 and 5 numerous theorems on almost periodicity and regular almost periodicity, are proved under conditions on X and T which vary from theorem to theorem. In sections 7 and 8 recurrence and incompressibility are studied under the condition that T is generative, that is, T is abelian and is generated by some compact neighbourhood of the identity. It is shown that the recurrence properties are essentially equivalent to incompressibility properties.

The group T is said to be transitive at x if for every open set U of X there exists $t \in T$ such that $xt \in U$. The

group is said to be regionally transitive if for every two open sets U and V of X there exists $t \in T$ such that $Ut \cap V \neq \emptyset$. Transitivity is studied in section 9. In section 10 asymptoticity is studied under the condition that X is a compact metric space and T is the cyclic group generated by a homeomorphism ϕ of X onto itself. A point x is said to be (positively) asymptotic to a closed invariant set B if $x \in B$ and $\lim \varrho(x\phi^n, B)=0$. Section 11 deals with function spaces with a view to the properties needed elsewhere.

Part I is written in a very precise terse style, and definitions and theorems are given under most general conditions. It contains a fully organized presentation of topological dynamics from a certain point of view.

Part II consists of three sections dealing with some specific examples of flows and has a much more classical approach to the subject. Sections 12 and 13 are respectively on symbolic dynamics and geodesic flows, subjects with which one of the authors has been intimately associated. The theorem on the topological transitivity of a geodesic flow on surfaces of constant negative curvature is proved. Section 14 is on cylinder homeomorphisms.

The book is a considerable technical achievement for the authors in their efficient and neat organization of the material. The reviewer regrets however the restrictive aims of the authors which lead to the omission of the mention of some recent progress in the theory where essentially new techniques were introduced, for example, the Gelfand and Fomin approach to the study of geodesic flows [Uspehi Mat. Nauk (N.S.) **7** (1952), no. 1(47), 118–137; MR **14**, 660]. The reviewer also found the reading of the book heavy going because of the brevity and condensation of the style. *Y. N. Dowker* (London).

Słowikowski, W. Note on the general theory of closed continuous mappings of bicompact T_1 spaces. Bull. Acad. Polon. Sci. Cl. III. 3 (1955), 421–424.

"In this paper, I wish to present an algebraic point of view on the theory of closed continuous mappings of bicompact T_1-spaces. In this way I obtain the natural characterisation of the family of inverse images of an arbitrary closed continuous mapping of bicompact T_1-spaces. This result may be applied to the theory of semicontinuous decompositions of bicompact T_1-spaces." (Author's summary.)

M. Henriksen (Lafayette, Ind.).

Iséki, Kiyoshi. On almost periodic transformations on uniform spaces. Proc. Japan Acad. 31 (1955), 340.

Let X be a uniform space and let f be a continuous map of X into X. The map f is said to be positively almost periodic provided that for each surrounding U of X there exists a positive integer N such that every set of N consecutive positive integers has an element n such that $(x, f^n(x)) \in U$ for all $x \in X$. The map f is said to be positively recurrent provided that for each surrounding U of X there exist infinitely many positive integers n such that $(x, f^n(x)) \in U$ for all $x \in X$. The following three theorems are announced without proof: (1) If f is uniformly continuous and positively recurrent, then so also if f^k for every positive integer k. (2) If f is uniformly continuous and positively almost periodic, then so also is f^k for every positive integer k. (3) If X is precompact, if f is a homeomorphism of X onto X, and if the set of all negative powers of f is uniformly equicontinuous, then f is positively almost periodic. *W. H. Gottschalk.*

Ward, L. E., Jr. Continua invariant under monotone transformations. J. London Math. Soc. 31 (1956), 114–119.

The author generalizes some theorems of Kelley [Duke. Math. J. **5** (1939), 535–537; MR **1**, 29] and Schweigert [Amer. J. Math. **66** (1944), 229–244; MR **5**, 274] on invariant sets. The improved results are:

Theorem I. If X is a continuum and $f(X)=X$ is monotone, then X contains a non-null subcontinuum Y which is minimal with respect to being f-invariant and having $f|Y$ monotone. Further, Y has no cutpoints.

Theorem II. Let X be a continuum, E an end element containing no cutpoints of X, and let $f(X)=X$ be monotone with $f(E)=E$. Then $X-E$ contains a non-null invariant subcontinuum without cutpoints.

Theorem III. Let X be a continuum, E an end element containing at least one cutpoint of X, $f(X)\subset X$ a monotone function such that $f(E)\subset E$ and for each positive integer n, $f(X)\cap f^{-n}(\overline{X-E})$ is a non-null connected set. Then either $\overline{X-E}$ contains a non-null f-invariant subcontinuum without cutpoints or E contains a fixed cutpoint of X.

Haskell Cohen (Baton Rouge, La.).

Cassina, Ugo. Elementi della teoria degli insiemi. I. Insiemi connessi. Period. Mat. (4) 33 (1955), 193–214.

The first of a series of expository articles on point-set topology in Euclidean space for the lay reader. The objective of the series is to characterize an arc. This first paper is on connected sets. Symbolism is kept to a minimum and historical background indicated.

M. E. Shanks (Lafayette, Ind.).

Scorza Dragoni, Giuseppe. Sulla teoria della dimensione. Matematiche, Catania **10** (1955), 104–120.

Expository lecture.

Weier, Josef. Über Zerlegung eindimensionaler Nullstellengebilde. Math. Z. **64** (1955), 115–122 (1956).

The author continues his study of 'finite deformations' [J. Math. Pures Appl. (9) **34** (1955), 137–143; MR **17**, 181 (we use the terminology of this review)]. He obtains a dissection theorem for singularities which can be stated roughly as follows: Let S be a singularity of the finite deformation F of index n and a_i, $1\leqq i\leqq k$, be integers whose sum is n. Then F is homotopic to a finite deformation G coinciding with F outside of a prescribed neighborhood of S and possessing in this neighborhood k singularities S_i, $1\leqq i\leqq k$, with index of S_i equal to a_i. *S. Stein.*

Ringel, Gerhard. Teilungen der Ebene durch Geraden oder topologische Geraden. Math. Z. **64** (1955), 79–102 (1956).

It is known that m lines, intersecting in $\binom{m}{2}$ distinct points, decompose the real affine plane into $\binom{m+1}{2}+1$ regions, of which $2m$ are infinite [J. Steiner, J. Reine Angew. Math. **1** (1826), 349–364]. The arrangement is topologically unique when $m\leqq 4$, but occurs in 6 forms for $m=5$, 43 for $m=6$, 922 for $m=7$ [R. Klee, Über die einfachen Konfigurationen der euklidischen und projektiven Ebene, Focken-Oltmanns, Dresden, 1938]. For larger values of m the classification is facilitated, though the number of possibilities is increased, by allowing the lines to be curved ("Pseudogeraden") while still insisting that each pair meet just once. Each line decomposes the

plane into two half-planes which are arbitrarily labelled + and −. Every region is described by a row of m signs specifying the half-plane in which it lies with respect to each line in turn. Every arrangement is described by a matrix formed by such rows of signs, one for each region; but of course the arrangement is not affected by any permutation of rows or of columns, nor by the reversal of signs in a whole column. The author obtains necessary and sufficient conditions for such a matrix to be geometrically realizable. He admits that more stringent conditions may be required in order to straighten the lines. For instance, the concurrences in the Pappus configuration can be destroyed by small variations, but the number of ways of doing this is smaller if the lines remain straight than if they are allowed to bend. *H. S. M. Coxeter.*

Berstein, I. Remarques sur un théorème de F. J. Dyson. Com. Acad. R. P. Romîne 5 (1955), 969–971. (Romanian. Russian and French summaries)

The following theorem is given. Suppose that T is an involution without fixed points on a locally connected, unicoherent continuum E. Let $\delta=\inf \varrho(x, Tx)$, and let f be a continuous real-valued function on E. If $0<d\leqq\delta$, there exist $a, b \in E$ with $\varrho(a, b)=d$ and $f(a)=f(b)=f(Ta)=f(Tb)$. This is a slight generalization of Livesay's theorem [Ann. of Math. (2) 59 (1954), 227–229; MR 15, 548], which in turn generalized the theorem of Dyson. *E. E. Floyd* (Charlottesville, Va.).

See also: Hoo and Chen, p. 570; Gel'fand, p. 575; Cesari, p. 596; Kowalsky, p. 642.

Algebraic Topology

★ **Postnikov, M. M. Issledovaniya po gomotopičeskoĭ teorii nepreryvnyh otobraženiĭ. I. Algebraičeskaya teoriya sistem. II. Natural'naya sistema i gomotopičeskiĭ tip. [Investigations in homotopy theory of continuous mappings. I. The algebraic theory of systems. II. The natural system and homotopy type.]** Trudy Mat. Inst. Steklov. no. 46. Izdat. Akad. Nauk SSSR, Moscow, 1955. 158 pp. 7.35 rubles.

The present paper is a revised and expanded version of the first two of the author's three short papers published in 1951 [Dokl. Akad. Nauk SSSR (N.S.), 76 (1951), 359–362, 789–791; MR 13, 374, 375], in which he gave a solution of the problems of classifying polyhedra by homotopy type and of classifying maps of polyhedra. Only the first problem is treated here; the second is promised later.

In providing an algebraic characterization of homotopy type the author shows explicitly how to compute the singular homology and cohomology groups from the homotopy groups together with the Postnikov invariants. The author expresses these invariants, as in the earlier papers, by means of a "natural system", but the definition of this system has been considerably simplified in the present version by the use of the Eilenberg-Zilber notions of semi-simplicial and minimal complexes [Ann. of Math. (2) 51 (1950), 499–513; MR 11, 734] to replace the author's normal complexes.

An introduction gives the basic facts of homotopy theory and singular homology theory so that the book can, in principle, be read without previous experience of the subject. The introduction also contains the results of Eilenberg and Maclane on the relations between homotopy and homology groups [Ann. of Math. (2) 46 (1945), 480–509; 51 (1950), 514–533; MR 7, 137; 11, 735].

The main body of the work is divided into two parts, the first part being purely algebraical, and largely independent of the introduction. In this part the notion of semi-simplicial complexes is introduced and developed to advantage. These is some departure from Eilenberg-Zilber in the author's treatment of cohomology with local coefficients in a semi-simplicial complex.

The second part presents a detailed exposition of singular cohomology in a topological space. Here the author explicitly adopts the Eilenberg-Zilber definition of minimal complexes but continues, as in the earlier version, to use the terminology 'normal' complexes. The classification theorem for cell-polyhedra (CW-complexes in the sense of J. H. C. Whitehead) is stated and proved and several known results are deduced explicitly. The book ends with an appendix proving, as a ready consequence of the author's results, Serre's theorem as the finiteness of the number of generators of a simply-connected space with finitely-generated singular homology groups. *P. J. Hilton* (Manchester).

Guggenheimer, Heinrich. Sulla teoria globale delle trasformazioni puntuali. Boll. Un. Mat. Ital. (3) 10 (1955), 474–477.

A continuous correspondence $T(\alpha, \beta)$ (where α, β are positive integers) between two topological spaces X, Y is defined to be a mapping of X onto Y such that 1) there exist subspaces $A\subset X$, $B\subset Y$ such that to every point in $X-A$ there correspond, under T, α points in $Y-B$ and to every point in $Y-B$ there correspond, under T^{-1}, β points in $X-A$; 2) $T|X-A$, $T^{-1}|Y-B$ are continuous; 3) $\dim A<\dim X$, $\dim B<\dim Y$ if the dimensions are finite. It is shown how special correspondences of the form $T(1, 1)$ between projective spaces lead to various facts concerning the homological and homotopical structures of the sets A and B. *P. V. Reichelderfer.*

Borsuk, K. Sur la notion de dépendance des transformations continues. Bull. Acad. Polon. Sci. Cl. III. 3 (1955), 251–254.

Let A be a compact space, and Y a compact ANR. Denote by Y^A the space of maps of A into Y. Notice that if A is a subspace of X, there is a natural map of Y^X into Y^A defined sending $f \in Y^X$ into $f|A \in Y^A$. Suppose that Φ is a subset of Y^A. A function $g \in Y^A$ is said to depend on Φ if for every compact space X containing A, $g \in$ image Y^X whenever $\Phi\subset$ image Y^X. Let $D(\Phi)$ denote the set of functions dependent on Φ. It is clear that if $g \in D(\Phi)$ then so does every element of the homotopy class of g.

The author says that $g \in Y^A$ depends on Φ in dimension m if for every compact space $X\supset A$ such that the dimension of $X-A$ is less than or equal to m, $\Phi\subset$ image Y^X implies $g \in$ image Y^X. Let $D_m(\Phi)$ denote the set of functions dependent on Φ in dimension m. We have $D(\Phi)\subset D_m(\Phi)$ and that if $m\leqq m'$, then $D_{m'}(\Phi)\subset D_m(\Phi)$. We may now state the main result of the author as follows: Theorem. If A is a compact space of dimension less than or equal to n, Y the n-dimensional sphere, m an integer, $n<m<2n$, and $\Phi=\{f_1, \cdots, f_k\}$ where $f_i \in Y^A$, then $D_m(\Phi)$ consists of those functions $g \in Y^A$ such that the homotopy class of g belongs to the subgroup of the Hopf group (n-dimensional cohomotopy group) of A generated by the homotopy classes of $f_1, \cdots, f_k$. *J. C. Moore.*

Hilton, P. J. Remark on the factorization of spaces. Bull. Acad. Polon. Sci. Cl. III. 3 (1955), 579–581.

The author gives a partial answer to a question of R. H. Fox [Fund. Math. **34** (1947), 278–287; MR **10**, 316]: If $X=Y\times Z$, do the homotopy types of X and Y determine that of Z? It is proved: If X is connected, has all its singular homology groups finitely generated, and belongs to the homotopy type of $Y\times Z$, then the homology of Z is completely determined by that of X and Y. If, in addition, X is simply connected, then also the homotopy groups of X and Y determine those of Z. *J. Dugundji.*

Spanier, E. H.; and Whitehead, J. H. C. Duality in homotopy theory. Mathematika 2 (1955), 56–80.

The Alexander duality theorem says that if X is a compact subset of the n-dimensional sphere, then $H_r(S^n-X)$, the r-dimensional reduced singular homology group of S^n-X, is isomorphic with $H^{n-r-1}(X)$, the $(n-r-1)$-dimensional reduced Čech cohomology group of X. The authors generalize this theorem in case X is a subpolyhedron of S^n.

For any space X let S^kX denote the k-fold suspension of X, i.e. $S^0X=1$, $S'X$ is the join of X with S^0 (a pair of points, and $S^{k+1}X$ is $S'S^kX$. If $f: X\to Y$ is a map, then let S^kf: $S^kX\to S^kY$ be the map induced by f. The set of homotopy classes of maps of S^kX into S^kY, denoted by $[S^kX, S^kY]$ is a group if $k\geqq 1$, and there is a natural homomorphism of $[S^kX, S^kY]$ into $[S^{k+1}X, S^{k+1}Y]$. In S theory, as developed by the authors in a paper in course of publication, one studies the direct limit group formed from the $[S^kX, S^kY]$ and the natural homomorphism $[S^kX, S^kY]$ into $[S^{k+1}X, S^{k+1}Y]$. This group is denoted by $\{X, Y\}$. If X, Y, Z are spaces, there is a natural map of $\{X, Y\}+\{Y, Z\}$ into $\{X, Z\}$. Now $\alpha\in\{X, Y\}$ is defined to be an S-equivalence if and only if there exists $\alpha^{-1}\in\{Y, X\}$ such that $\alpha^{-1}\alpha$ is the element of $\{X, X\}$ represented by the identity map of X into X and $\alpha\alpha^{-1}$ is the element of $\{Y, Y\}$ represented by the identity map of Y into Y. If $A\subset X$, then A is an S-deformation retract of X if $\{i\}\in\{A,X\}$ is an S-equivalence where i: $A\to K$ is the inclusion map.

Suppose that X is a polyhedron in S^n. An n-dual of X is a polyhedron in S^n-X which is an S-deformation retract of S^n-X. The authors prove that if X^* is an n-dual of X, then X is an n-dual of X^*, thus justifying the definition. Let D_nX, denote an n-dual of X. One of the main theorems says that there is an isomorphism D_n: $\{X, Y\}\to\{D_nY, D_nX\}$.

Several applications of the general theory are given. One of these asserts that if X is a finite CW-complex of dimension less than or equal to $2i-2$, then $\pi^i(X)$ the i-dimensional cohomotopy group of X is finitely generated. *J. C. Moore* (Princeton, N.J.).

Inoue, Yoshiro. On exactness of the homotopy sequence of a p-ad. Math. Japon. 3 (1955), 97–102.

The homotopy groups of a p-ad and the homotopy sequence of a p-ad are natural generalizations of the homotopy groups and sequences of pairs and triads [see A. L. Blakers and W. S. Massey, Ann. of Math. (2) **53** (1951), 161–205; MR **12**, 435]. In this note the author gives a neat proof of the exactness of the homotopy sequence of a p-ad ($p\geqq 3$) by using the exactness of the homotopy sequence of a fibre space (in the sense of J. P. Serre). *W. S. Massey* (Providence, R.I.).

Čogošvili, G. S. On the application of direct spectra of bicompact groups in homology theory. Soobšč. Akad. Nauk Gruzin. SSR 15 (1954), 655–662. (Russian)

The author introduces several new homology and cohomology groups, associated with a pair (R, M), R locally compact normal, M a subset of R, by a limit construction. Let X be a discrete (abelian) group and θ its character group. Then $\nabla_I{}^r(M, R; X)$ is the limit group of the inverse system, whose generic group is, for any F (closed in R, contained in M), the factor group $H^r(F; X)-i^*H^r(R; X)$ (cohomology group of F modulo the image of the cohomology group of X under restriction to F), with the homorphisms defined in the obvious fashion. One also defines $\Delta_E{}^r(M, R; \theta)$ as the limit of the direct system, formed by the kernels of the injection of the homology groups of the same sets F as above into R; the groups are compact, and the limit is understood in the author's sense [Mat. Sb. N.S. **28(70)** (1951), 89–118; MR **12**, 846]. The two groups obtained are dual, in the author's sense. A similar pair of groups ($\nabla_E{}^r$ and $\Delta_I{}^r$) is defined by considering the directed set of open sets of R, containing M. If N is the complement of M in R, it is shown that

$$\nabla_I{}^r(M, R; X)\approx\nabla_E{}^{r+1}(N, R; X),$$
$$\Delta_E{}^r(M, R; \theta)\approx\Delta_I{}^{r+1}(N, R; \theta).$$

This generalizes known duality laws for closed sets [e.g., P. S. Aleksandrov, Izv. Akad. Nauk SSSR. Ser. Mat. **6** (1942), 227–282; MR **4**, 249]. The second part considers spectral-singular homology groups, as discussed by Hurewicz, Dugundji, and Dowker [Ann. of Math. (2) **49** (1948), 391–406; MR **9**, 606] and the author [Soobšč. Akad. Nauk Gruzin. SSR **14** (1953), 583–588; MR **16**, 389]. A boundary operator is defined from the relative groups to those of the subset; it is stated and partially proved that the Eilenberg-Steenrod axioms hold. *H. Samelson* (Ann Arbor, Mich.).

Jaworowski, J. W. Involutions of compact spaces and a generalization of Borsuk's theorem on antipodes. Bull. Acad. Polon. Sci. Cl. III. 3 (1955), 289–292.

M is a compactum which is Čech-acyclic in all dimensions below n and admits a continuous involution p. If f is continuous on M into S^n so that $fpx\neq fx$ for all $x\in M$, then the induced homomorphism f_* is onto for $H_n(M, G_2)\to H_n(S^n, G_2)$. An indirect consequence is that any continuous involution of a compactum acyclic mod 2, has a fixed point. *D. G. Bourgin* (Rome).

Jaworowski, J. W. On some properties of mappings of the sphere into the Euclidean space. Bull. Acad. Polon. Sci. Cl. III. 3 (1955), 583–584.

The author looks at a problem of finding a continuous mapping from an n-sphere into an euclidean $(n-p)$-space such that the set of points whose images under the mapping are the same as those of the antipodals pointwisely, coincide with a given closed antipodal set on the sphere. He gives the answer for the case $p=n-1$ and $p=0$ by constructing the mapping. *H. Yamabe.*

Massey, W. S. Some problems in algebraic topology and the theory of fibre bundles. Ann. of Math. (2) 62 (1955), 327–359.

Le Colloque sur la géométrie différentielle et les espaces fibrés tenu à Cornell University (mai 1953) a donné lieu à deux rapports détaillés. Le premier, dû à F. Hirzebruch, est consacré aux problèmes concernant la géométrie différentielle et la théorie des variétés analytiques complexes

[Ann. of Math. (2) **60** (1954), 213–236; MR **16**, 518]. Le second est le présent mémoire; il traite à la fois de la topologie algébrique et des espaces fibrés, car ces deux théories sont devenues étroitement mêlées. Une liste de 48 problèmes est répartie sous les rubriques suivantes: 1) définitions diverses d'un espace fibré; 2) suites spectrales; 3) groupes d'homotopie (méthodes générales de calcul, groupes d'homotopie des sphères, opérations homotopiques, type d'homotopie, espaces d'Eilenberg-MacLane, opérations cohomologiques, classification des applications et obstructions, groupes de cohomotopie); 4) homologie des groupes de Lie et de leurs espaces homogènes; 5) H-espaces.

L'article contient une bibliographie détaillée pour les années récentes, et un Appendice consacré à l'état actuel des 46 problèmes de Topologie mentionnés par Eilenberg [ibid. **50** (1949), 247–260; MR **10**, 727]; une vingtaine ont été résolus en totalité ou en partie. *H. Cartan* (Paris).

Švarc, A. S. Homologies of the spinor group. Dokl. Akad. Nauk SSSR (N.S.) **104** (1955), 26–29. (Russian)

The author announces the result that the algebra $H^*(\mathrm{Spin}\,(n), Z_2)$ has a primitive system of generators if and only if $n\leqq 9$ or $n=2^\alpha+1$ (α positive integer), and the Pontrjagin ring of Spin (n) mod 2 is a Grassmann algebra if and only if $n\leqq 9$ or $n=2^\alpha+1$. The statement for $n=2^\alpha+1$ contradicts an announcement of Borel [Amer. J. Math. **76** (1954), 273–342, p. 326; MR **16**, 219].

The main tool in proving the present results is the twofold covering of the Whitehead-Miller complex which has been employed also by Araki [Mem. Fac. Sci. Kyûsyû Univ. Ser. A. **9** (1955), 1–35; MR **17**, 11]. The reviewer has not been able to check all the details since many of them are only indicated. Further, the author describes a class of primitive generators in the weak homology of Spin (n) which is connected with a similar system for SO(n) described by Dynkin [Uspehi Mat. Nauk (N.S.) **8** (1953), no. 5(57), 73–120; MR **15**, 601].

W. T. van Est (Utrecht).

Vázquez García, R. Cup-i products of cochains in cubical singular cohomology theory. Bol. Soc. Mat. Mexicana **11** (1954), 9–32. (Spanish)

J.-P. Serre has given an explicit formula for the cup product of cochains in cubical singular cohomology theory [Ann. of Math. (2) **54** (1951), 425–505; MR **13**, 574]. It was proved by S. MacLane [Proc. Amer. Math. Soc. **5** (1954), 642–651; MR **16**, 160] that this definition of Serre gives rise to the same product of cohomology classes as that given by the standard definition for the product of singular simplicial cochains. In this paper the author gives an explicit formula for the Steenrod cup-i products of singular cubical cochains. He then proves that the Steenrod squares modulo two defined by means of these products satisfy the axioms of Serre [Comment. Math. Helv. **27** (1953), 198–232; MR **15**, 643] which uniquely characterize the Steenrod squares (mod 2).

W. S. Massey (Providence, R.I.).

Inoue, Yoshiro. On singular cross sections. Proc. Japan Acad. **31** (1955), 678–681.

The problem of constructing cross-sections in a fibre space is treated in terms of singular complexes. The extension theorems which apply in the case where the base space is a simplicial complex are generalized. No proofs are given. There are applications to the case of fibre spaces over a CW-complex. A different approach to this case has been made by Barcus [Quart. J. Math. Oxford Ser. (2) **5** (1954), 150–160; MR **16**, 160].

I. M. James (Princeton, N.J.).

Borsuk, K.; and Kosiński, A. Families of acyclic compacta in euclidean n-space. Bull. Acad. Polon. Sci. Cl. III. **3** (1955), 293–296.

Let X be a compactum and Y metric separable. In particular, $Y=E_n$, the Euclidean n-space. Let Φ be upper semi-continuous on $X\rightarrow 2^Y$, where 2^Y refers to the non-empty compact subsets of Y. $F=(X, Y, \Phi)$ is called a family. If $\Phi_2(x)\subset\Phi_1(x)$, then $F(\ ,\Phi_1)$ prolongs $F(\ ,\Phi_2)$. Denote by $S(F)$ the family of all prolongations $\{F(X, E_n, \psi)\}$ for which $\psi(X)$ includes all bounded components of $E_n-\Phi(X)$. Let G_m be the group of integers mod m and let $j:\Phi_2(X)\rightarrow\Phi_1(X)$ be the inclusion map. Theorem: If the induced homorphism on $H_{n-1}(\Phi_2(X), G_m)\rightarrow H_{n-1}(\Phi_1(X), G_m)$ is trivial, then $F_1=F(\ ,\Phi_1)\in S(F_2)$. Write $P_m(F)$ for all prolongations whose set $\psi(x)$ are acyclic for the coefficient groups G_m. Suppose (a) $\Phi(x)\cap\Phi(x')=\emptyset$ $(x\neq x')$. Then, Theorem: If the homomorphism h induced by Φ^{-1} on $H_{n-1}(\Phi(X), G_m)\rightarrow H_{n-1}(X, G_m)$ is trivial, then $P_m(F)\subset S(F)$. (A similar result is obtained even when (a) is dropped). Suppose $F_1\in P_m(F)$. The proof pivots on passing from $\Phi(X)$ and $\Phi_1(X)$ to the graphs of F and F_1 and so expressing the homomorphism h in terms of a chain of homomorphisms to which the Vietoris-Begle theorem [Begle, Ann. of Math. (2) **51** (1950), 534–543; MR **11**, 677] is applicable. Applications are indicated. *D. G. Bourgin* (Rome).

Kosiński, A. On manifolds and r-spaces. Fund. Math. **42** (1955), 111–124.

An r-space is defined to be a finite-dimensional, compact metric space such that if $p\in K$, then p has arbitrarily small neighborhoods ("canonical neighborhoods") U such that if $q\in U$, then $F(U)$ is a deformation retract of $U-q$. ($F(U)$ denotes the boundary of U.) The general problem is to find how many of the properties of a topological manifold are possessed by an r-space. A connected r-space of dimension $\leqq 2$ is already a topological manifold, but whether the analogue holds for higher dimensions is unsolved. Every connected r-space is a peanian Cantor-manifold. If K is an n-dimensional connected r-space, then a necessary and sufficient condition that a closed subset E of K be n-dimensional is that E have a non-empty subset which is open in K; and if U is a canonical neighborhood, $F(U)$ is an $(n-1)$-dimensional continuum. Special consideration is given to triangulable r-spaces, which are called r-polytopes. Every vertex star of an n-dimensional connected r-polytope has a boundary having the same homotopy type as the $(n-1)$-sphere. Connections are made with the "spheroidal spaces" of K. Borsuk [Mat. Sb. N.S. **1(43)** (1936), 643–660]. A necessary and sufficient condition that an n-dimensional r-polytope K be a spheroidal polytope is that K be of the homotopy type of the n-sphere. A connected polytope K of dimension <4 is a manifold if and only if it is an r-space; and is a spheroidal polytope if and only if it is an r-space and has the homotopy type of the sphere. A list of unsolved problems is given; e.g., does there exist an r-polytope which is not a manifold? *R. L. Wilder.*

GEOMETRY

Čeněk, Gabriel. Remark on construction of the illumination of a spherical surface by orthogonal axonometric projection. Mat.-Fyz. Časopis. Slovensk. Akad. Vied 5 (1955), 152–161. (Slovak. Russian summary)

Goldberg, Michael. Basic rotors in spherical polygons. J. Math. Phys. 34 (1956), 322–327.

Continuing his work on rotors in spherical polygons [same J. 30 (1952), 235–244; MR 13, 577], the author constructs two types of "basic" rotor for an n-gon, as follows. Consider, on a sphere, a fixed circle of circumference c. Let a circle of circumference $nc/(n\pm 1)$, constrained to move on the sphere, roll within or about the fixed circle, as in the formation of an $(n+1)$-cusped hypocycloid or of an $(n-1)$-cusped pericycloid. The basic rotor is the envelope of a great circle, rigidly attached to the rolling circle in such a position that its pole is just near enough to the centre of the rolling circle for the envelope to be convex. *H. S. M. Coxeter* (Toronto, Ont.).

Rodeja F., E. G. Area of the ellipse determined by five tangents. Collect. Math. 7 (1954), 113–119.

Five lines, suitably situated in the real affine plane, determine a unique ellipse. The author obtains an expression for the area of this ellipse in terms of the areas of the ten triangles formed by the five lines. The expression is not symmetrical, and involves square roots whose signs are not discussed. *H. S. M. Coxeter.*

Kotzig, Anton. Contribution to the theory of Eulerian polyhedra. Mat.-Fyz. Časopis. Slovensk. Akad. Vied 5 (1955), 101–113. (Slovak. Russian summary)

It is impossible for every face of a Eulerian polyhedron to have more than 5 sides [L. Euler, Opera omnia, ser. 1, opera mathematica, vol. 26, commentationes geometricae, vol. 1, Soc. Sci. Nat. Helv., Lausanne, 1953, p. 88, Prop. 7; MR 15, 770]. Extending this theorem, the author proves that it is impossible for every pair of adjacent faces to have a total of more than 13 sides. If there are no triangular faces, the number 13 can be replaced by 11. That these results are best possible may be seen by considering the truncated dodecahedron, where triangles meet decagons, and the truncated icosahedron, where pentagons meet hexagons. *H. S. M. Coxeter* (Toronto, Ont.).

Metelka, Josef. Über ebene Konfigurationen $(12_4, 16_3)$. Časopis Pěst. Mat. 80 (1955), 133–145. (Czech. Russian and German summaries)

Continuing the work of Bydžovský [Věstník Královské České Společnosti Nauk. Třída Matemat.-Přírodověd 1939 no. 2; MR 7, 390], the author describes a method for classifying the fifty-odd known configurations $(12_4, 16_3)$. Numbering the points from 1 to 12, he considers cases where none of the six joins of the four points 9, 10, 11, 12 belongs to the configuration. His scheme shows which other points lie on lines through each of these four. In this manner he distinguishes eight such configurations, four of which are new. *H. S. M. Coxeter.*

Metelka, Václav. Über gewisse ebene Konfigurationen $(12_4, 16_3)$, welche mindestens einen D-Punkt Enthalten. Časopis Pěst. Mat. 80 (1955), 146–151. (Czech. Russian and German summaries)

The author describes two new configurations $(12_4, 16_3)$ whose twelve points all have rational coordinates. *H. S. M. Coxeter* (Toronto, Ont.).

Meynieux, R. Configuration de Milne, Dixon et Morton en géométrie de caractéristique 3 ou 2. Publ. Sci. Univ. Alger. Sér. A. 1 (1954), 295–302 (1955).

In his earlier paper on the vertices of the Steiner trihedra of a cubic surface [Publ. Sci. Univ. Alger. Sér. A. 1 (1954), 183–195; MR 16, 1143] the author set aside the cases where the underlying field of the geometry is of characteristic 2 or 3. He now fills this gap, observing that, in the case of characteristic 2, the Steiner trihedra are degenerate, each consisting of three planes through a line [cf. Coxeter, Amer. J. Math. 62 (1940), 457–486, p. 478; MR 2, 10]. *H. S. M. Coxeter.*

Crowe, D. W. The n-dimensional cube and the tower of Hanoi. Amer. Math. Monthly 63 (1956), 29–30.

The author shows that the possible positions in the puzzle called the Tower of Hanoi can be represented by the vertices of an n-dimensional cube. He proves that the sequence of positions corresponding to the solution of the problem defines a Hamiltonian circuit of the n-cube. *W. T. Tutte* (Toronto, Ont.).

Guggenheimer, H. Sur les axiomes de la géométrie plane. I. Riveon Lematematika 9 (1955), 49–53. (Hebrew. French summary)

This is the first of a series of papers on the foundations of geometry. The starting point is a set M of points and the relation 'between', denoted by (ABC), which is defined for some sets consisting of three different points of M. This relation is used to define: segments, angles, lines, etc. The author starts from 12 axioms which include the axioms of order and of congruence as given by Hilbert and he proves their independence. The formalism of mathematical logic is used in the definitions and in the list of axioms. *S. A. Amitsur* (Jerusalem).

★ **Širokov, P. A. Kratkiĭ očerk osnov geometrii Lobačevskogo. [A brief outline of the elements of Lobačevsian geometry.]** Gosudarstv. Izdat. Tehn.-Teor. Lit., Moscow, 1955. 87 pp. 1.20 rubles.

Tresse, A. Théorie élémentaire des géométries non euclidiennes. II. Bull. Soc. Math. France 83 (1955), 1–56.

This is the second part of a purely expository and elementary exposition (accessible to the high-school level in France) of hyperbolic geometry [for part I see same Bull. 81 (1953), 81–143; MR 15, 246]. In this part length and hyperbolic trigonometry are developed from the Poincaré model. An unusual feature is the discussion of pentagons with all right angles. *H. Busemann.*

See also: Ghika, p. 645; Ringel, p. 651; Kotzig, p. 655.

Convex Domains, Extremal Problems

Sherman, S. A theorem on convex sets with applications. Ann. Math. Statist. 26 (1955), 763–767.

In one dimension if two distributions are unimodal and symmetrical then so also is their convolution. The author obtains a kind of generalization of this to distributions in n-dimensional Euclidean space $\mathcal{E}$; his generalization covers a partial result obtained by T. W. Anderson [Proc. Amer. Math. Soc. 6 (1955), 170–176, Th. 1; MR 16,

1005]. The author introduces the closed convex cone C_1 generated in the L_1 norm (that is, $\|f\|_1=\int|f(x)|dx$) by the characteristic functions of symmetrical, compact, convex sets, and also the cone C_3 obtained similarly with the norm max $(\|f\|_1, \max|f(x)|)$. When $n=1$ functions in C_1 or C_3 are (bounded) densities of symmetrical unimodal continuous distributions. He proves that (Theorem 2) $C_1*C_1=C_1$ and $C_3*C_3=C_3$. He applies this to generalize a lemma due to Z. W. Birnbaum [Ann. Math. Statist. 19 (1948), 76–81; MR 9, 452] on "peakedness". The author calls a function $\phi(x)$ more peaked about the origin O than $f(x)$ when the convolution $(\phi-f)*\chi_E$ is non-negative at O whenever χ_E is the characteristic function of a compact, symmetrical, convex set E in $\mathcal{E}$. [Remark: the author's Theorem 2 would be more useful if it could be supplemented by convenient sufficient conditions for a function to belong to C_1 (or C_3).] *H. P. Mulholland.*

Wu, Teh-Tau. On ovals of n-type. Acta Math. Sinica 3, 213–217 (1953). (Chinese. English summary)

Let $\varrho(\varphi)$ be the radius of curvature of an oval C at a generic point and let

$$a_k=\frac{1}{\pi}\int_0^{2\pi}\varrho(\varphi)\cos k\varphi\,d\varphi,\quad b_k=\frac{1}{\pi}\int_0^{2\pi}\varrho(\varphi)\sin k\varphi\,d\varphi \qquad (k=1,2,\cdots),$$

be the Fourier coefficients. C is said to be of type n if $a_1=\cdots=a_n=0$, $b_1=\cdots=b_n=0$, but at least one of a_{n+1} and b_{n+1} is different from zero. The following theorems are proved. 1) An oval of type $2(2m+1)$ has at least $2m+3$ squares circumscribed to it, $m=0, 1, 2, \cdots$. 2) On an oval of type n there are at least $n+1$ or $n+2$ pairs of opposite points, according as n is odd or even, such that the sum of the radii of curvature at each pair is an extremum. *S. Chern* (Chicago, Ill.).

See also: Mahler, p. 589; Cesari, p. 596; Rodeja F., p. 655; Vidal Abascal, p. 656; Santaló, p. 656.

Differential Geometry

★ **Vidal Abascal, E. Introduccion a la geometria diferencial. [Introduction to differential geometry.]** Editorial Dossat, S.A., Madrid, 1956. xvi+329 pp. 220 ptas.

The book covers the standard material of a course in differential geometry with several notable additions and variations. The theory is first treated by vector methods, later in Chapter V exterior forms and differentiation are introduced, and the principal results are briefly derived again by these methods.

Besides the introduction, which treats vectors and matrices in detail there are six chapters I, $\cdots$, VI. We list the chapter headings and give some details in parentheses: I. Plane curves (includes curves of constant width). II. Space curves (with a detailed discussion of the surfaces enveloped by the osculating, normal and rectifying plane). III. Elementary theory of surfaces (first and second fundamental forms). IV. The fundamental equations of surface theory (equations of Gauss, Weingarten, Codazzi-Mainardi, lines of curvature, asymptotic lines). V. Geometry on a surface (besides the already mentioned use of exterior differentiation, an introduction to the integral geometry on surfaces, in particular on surfaces of constant curvature, deserves mention as an unusual feature). VI. Representation of surfaces, special surfaces (differential parameters, conformal and other mappings, minimal surfaces, ruled surfaces, line congruences).

The book is easy to read, but leaves something to be desired regarding rigour. For example, the non-obvious existence of normal coordinates of class $\geqq 2$ is taken for granted. *H. Busemann* (Los Angeles, Calif.).

Santaló, L. A. Questions of differential and integral geometry in spaces of constant curvature. Univ. e Politec. Torino. Rend. Sem. Mat. 14 (1954–55), 277–295. (Spanish)

For the special case of Riemannian spaces of constant curvature, K, the author has simplified the statement of the generalized Gauss-Bonnet Theorem. Let Q be an n-dimensional set, in a space of constant curvature, bounded by a surface S. Let V be the volume of Q, F the area of S, M_i $(i=1, \cdots, n-1)$ the integrals over S of the mean curvatures of S, O_i the area of the unit sphere of dimension i,

$$c_i=\binom{n-1}{i}\frac{O_n}{O_iO_{n-1-i}}K^{(n-1-i)/2}$$

and $\chi(Q)$ the Euler-Poincaré characteristic of Q.

Then the Gauss-Bonnet Theorem takes the form:

$$c_{n-1}M_{n-1}+\cdots+c_1M_1+K^{n/2}V=\tfrac{1}{2}O_n\chi(Q) \text{ for } n \text{ even},$$

$$c_{n-1}M_{n-1}+\cdots+c_2M_2+c_0F=\tfrac{1}{2}O_n\chi(Q) \text{ for } n \text{ odd}.$$

In the derivation of these results, interpretations are given for the differential forms Φ_h and Ψ_h defined by Chern. Applications are made to various questions in integral geometry, in which formulas previously known for lower dimensions or for Euclidean spaces only are extended to general spaces of constant curvature. *C. B. Allendoerfer* (Seattle, Wash.).

Nádeník, Zbyněk. Les surfaces analogiques aux courbes de Bertrand. Czechoslovak Math. J. 5(80) (1955), 194–219. (Russian. French summary)

Verfasser betrachtet Flächenpaare $\mathfrak{r}(u,v)$ und $\mathfrak{r}^*(u,v)$ im dreidimensionalen euklidischen Raum und auf beiden Paare orthogonaler Kurvenkongruenzen O_1, O_2 auf $\mathfrak{r}$ und O_1^*, O_2^* auf $\mathfrak{r}^*$. Die zugehörigen Tangentenvektoren seien $\mathfrak{t}_1$, $\mathfrak{t}_2$ und $\mathfrak{t}_1^*$, $\mathfrak{t}_2^*$ und die Flächen-normalenvektoren $\mathfrak{n}$ und $\mathfrak{n}^*$. Gesucht wird eine eineindeutige Abbildung C der beiden Flächen, derart daß erstens die Normalen in $\mathfrak{r}$ und $\mathfrak{r}^*=C\mathfrak{r}$ nicht derselben Ebene angehören, zweitens die Beziehungen

$$\mathfrak{r}^*=\mathfrak{r}+\varrho_1\mathfrak{t}_1+\varrho_2\mathfrak{t}_2+\varrho_3\mathfrak{n},\qquad \begin{aligned}\mathfrak{t}_1^*&=\lambda_1\mathfrak{t}_1+\lambda_2\mathfrak{t}_2+\lambda_3\mathfrak{n}\\ \mathfrak{t}_2^*&=\mu_1\mathfrak{t}_1+\mu_2\mathfrak{t}_2+\mu_3\mathfrak{n}\\ \mathfrak{n}^*&=\nu_1\mathfrak{t}_1+\nu_2\mathfrak{t}_2+\nu_3\mathfrak{n}\end{aligned}$$

bestehen für jedes entsprechende Punktepaar der Abbildung. Dabei sing ϱ_i, λ_i, μ_i, ν_i $(i=1,2,3)$ Konstante und die Matrix $(\lambda\,\mu\,\nu)$ ist orthogonal (Analogie zu Bertrand's Kurvenpaaren). Lösungen des Problems heißen B-Flächen, die zugehörigen Kongruenzen B-Kongruenzen und $\mathfrak{r}^*$ assoziiert zu $\mathfrak{r}$. Sind f_1, a, b bzw. geodätische Krümmung, Normalkrümmung und geodätische Torsion einer Kurve aus O_1 und analog f_2, c, $-b$ aus O_2, so ist $\mathfrak{r}$ dann und nur dann eine B-Fläche, wenn

$$f_1=pa+qb+r_1,\quad f_2=pb+qc+r_2,\qquad p, q, r_1, r_2 \text{ Konstante},\ r_1^2+r_2^2\neq 0.$$

Das Analogon zur bekannten Bertrandschen Beziehung

zwischen Krümmung und Torsion einer Bertrandschen Kurve lautet für eine Bertrandsche Fläche

$$(p^2+q^2+1)K+(pr_1+qr_2)H+r_1^2+r_2^2=0.$$

Weiterhin werden spezielle Eigenschaften von B-Flächen studiert, insbesondere solche abwickelbarer B-Flächen.

M. Pinl (Köln).

Alguneid, Ali Rida. A geometrical illustration of the destruction of loci by degenerate collineations. Proc. Egyptian Acad. Sci. **10** (1954), 69–72 (1955).

Continuation of a paper on degeneration of space collineations [same Proc. **7** (1951), 1–17; MR **14**, 681]. A surface S given as a locus, an envelope of planes and a complex of tangent lines is by a "complete" collineation transformed into S'. The author considers the limiting form of S' when the transformation becomes degenerate. Examples. *O. Bottema* (Delft).

Ancochea, Germán. On the geometric interpretation of the projective curvature of a real plane curve. Abh. Math. Sem. Univ. Hamburg **20** (1955), 52–56. (Spanish)

Data una curva piana reale C, l'Autore determina in un generico punto A di C un riferimento intrinseco collegato alla cubica osculatrice ed alla curva di curvatura nella osculatrice in A alle C. Ciò permette all'Autore di definire la curvatura proeittiva e l'arco proiettivo e di assegnarne un elegante significato geometrico. *C. Longo* (Roma).

Souriau, Jean-Marie. Equations canoniques et géométrie symplectique. Pub. Sci. Univ. Alger. Sér. A. **1** (1954), 239–265 (1955).

The present paper is an elaboration of an earlier communication [Géométrie différentielle, Colloq. Internat. Centre Nat. Rech. Sci., Strasbourg, 1953, pp. 53–59; MR **15**, 648] concerning the application of symplectic differential geometry to the calculus of variations and the theory of partial differential equations of first order. It would lead too far to describe the author's complete theory of symplectic geometry, the latter being developed as a special case of a more general vector geometry. Certain types of subspaces, the so-called isotropic saturated submanifolds, seem to be particularly suitable for the description of the properties of characteristics and thus enable the author to construct a general theory — analogous to that of Jacobi — of first-order partial differential equations, canonical systems and related topics. It is suggested that this theory is of practical value also.

H. Rund (Toronto, Ont.).

Miron, Radu. Sur la torsion totale d'une surface. Gaz. Mat. Fiz. Ser. A. **7** (1955), 417–424. (Romanian. Russian and French summaries)

L'auteur introduit, pour une surface de l'espace ordinaire, un nouvel invariant qu'il appelle la torsion totale de la surface. Il fait des applications de la notion introduite à la caractérisation de certaines surfaces et de certaines courbes tracées sur elles, et présente sous une forme simple la formule classique d'O. Bonnet. Des possibilités d'extension des questions envisagées au cas des variétés non holonomes de l'espace euclidien à trois dimensions sont indiquées. *P. Vincensini* (Marseille).

Hsiung, Chuan-Chih. A theorem on surfaces with a closed boundary. Math. Z. **64** (1955), 41–46 (1956).

By means of integral and differential methods the author has succeeded in proving the following interesting theorem: Let S and S^* be two orientable surfaces of class C''' in a three-dimensional Euclidean space E^3 with positive Gaussian curvatures and closed boundaries C and C^* respectively. Suppose that there is a one-to-one correspondence between the points of the two surfaces S and S^* such that at corresponding points the two surfaces S and S^* have the same normal vectors and equal sums of the principal radii of curvature, and such that at corresponding points the two boundaries C and C^* have the same tangent vectors and equal linear elements. Then the two surfaces S and S^* are congruent or symmetric. *L. Auslander* (Princeton, N.J.).

Ahlfors, Lars V. Conformality with respect to Riemannian metrics. Ann. Acad. Sci. Fenn. Ser. A. I. no. **206** (1955), 22 pp.

Let S be a surface with a Riemannian metric $ds^2=Edx^2+2Fdxdy+Gdy^2$. Then the problem of introducing isothermal coordinates is that of finding coordinates such that $E=G$ and $F=0$. If we write this metric in the form $ds^2=E|dz+hd\bar{z}|^2$ with $dz=dx+idy$, the result of Korn and Lichtenstein states the existence of isothermal parameters if h satisfies a Hölder condition. The standard proof of this result consists of first introducing the isothermal parameters in the small and then using uniformization theory to get them in the large.

In the present paper the author obtains the isothermal coordinates (under the Kron and Lichtenstein criterion) in one step by solving a singular integral equation resulting from the Hilbert transform. The present proof is completely self-contained and contains a number of useful lemmas on the Hilbert transform and various spaces of functions satisfying a Hölder condition. These results are applied to obtain a variational theory for conformal mapping. *H. L. Royden* (Stanford, Calif.).

Chern, Shiing-shen. An elementary proof of the existence of isothermal parameters on a surface. Proc. Amer. Math. Soc. **6** (1955), 771–782.

In a domain D of the (x, y)-plane, suppose that the functions E, F, G satisfy a Hölder condition of order λ, $0<\lambda<1$. Then every point of D has a neighborhood whose local coördinates u, v are isothermal parameters, i.e., such that the metric $ds^2=Edx^2+2F\,dxdy+G\,dy^2$ takes the form $ds^2=\lambda(u, v)(du^2+dv^2)$, $\lambda(u, v)>0$. The proof of this theorem of Korn and Lichtenstein [L. Lichtenstein, Bull. Internat. Acad. Sci. Cracovie. Cl. Sci. Math. Nat. Sér. A. **1916**, 192–217] is effected by solving an integro-differential equation by successive approximations. Simplifications over the classical method occur through consistent use of complex notation. A similar proof is supposed to have been given by L. Bers in unpublished lecture notes (1951–1952); see also Vekua, Dokl. Akad. Nauk SSSR (N.S.) **100** (1955), 197–200; MR **16**, 1114]. The Hölder condition has been relaxed by Hartman and Wintner [Amer. J. Math. **77** (1955), 329–354; MR **17**, 627]. *L. W. Green* (Minneapolis, Minn.).

Chern, Shiing-shen. On special W-surfaces. Proc. Amer. Math. Soc. **6** (1955), 783–786.

Complex notation is employed to give a very brief proof of the theorem of H. Hopf: the only closed special analytic W-surfaces of genus zero are spheres [Math. Nachr. **4** (1951), 232–249; MR **12**, 634]. Hartman and Wintner [Amer. J. Math. **76** (1954), 502–508; MR **16**, 68] replaced the assumption of analyticity by the con-

dition that the surface be C^3-imbedded in Euclidean space. A W-surface (Weingarten surface) is called special if the relation between the principal curvatures can be written in the form $f(H, \mu)=0$, $\mu=H^2-K$, with $f_H\neq 0$ at every umbilic. (f was assumed to be of class C^3 by Hartman and Wintner; the author needs only C^1.) By an elegant treatment of the equation $w_{\bar{z}}=P(w\bar{w})_z+Qw\bar{w}$, it is proved that the umbilics of such a surface are isolated and have negative index. In view of the increasing importance of this class of surfaces, the author has subsequently suggested that a more suitable name is "'elliptic' W-surface'$\cdots$because the surfaces derive their properties mainly from the fact that their partial differential equation is elliptic in the neighborhood of an umbilic."
L. W. Green (Minneapolis, Minn.).

Čech, Eduard. Déformation ponctuelle des congruences de droites. Czechoslovak Math. J. **5(80)** (1955), 234–273. (Russian. French summary)

Verfasser betrachtet zwei m-dimensionale Punktmannigfaltigkeiten V_m und V_m' erzeugt durch die Punkte $x=x(u_1, u_2, \cdots, u_m)$, $x'=x'(u_1, u_2, \cdots, u_m)$. Sodann werden für jedes $u=(u_1, u_2, \cdots, u_m)$ Parameter $\alpha_1, \alpha_2, \cdots, \alpha_m$ willkürlich gewählt und zwischen den Tangentialräumen von V_m in x und V_m' in x' die Homographie

$$\mathbf{K}x=x',\ \mathbf{K}\frac{\partial x}{\partial u_i}=\frac{\partial x'}{\partial u_i}+\alpha_i x'\quad (i=1, 2, \cdots, m)$$

angesetzt. $\mathbf{K}$ heißt Tangentialhomographie; für jedes u gibt es deren ∞^m. Sind S_r und $S_{r'}$ zwei projektive Räume und gilt $V_m\subset S_r$, $V_m'\subset S_{r'}$, so hat $\mathbf{K}$ die charakteristische Eigenschaft, in eine Homographie zwischen den beiden projektiven Räumen fortsetzbar zu sein. Unter Kongruenz versteht Verfasser in dieser Abhandlung ∞^2 Geraden eines projektiven S_r, die nicht in einer festen Ebene enthalten sind. Wird eine solche Kongruenz L durch die Gerade $p=p(u, v)=(xy)$, $x=x(u, v)$, $y=y(u, v)$ erzeugt, so heißt der lineare Raum $\tau(u, v)$ bestimmt durch die Punkte x, y, $\partial x/\partial u$, $\partial x/\partial v$, $\partial y/\partial u$, $\partial y/\partial v$ Tangentialraum der Kongruenz L entlang der Geraden $p(u, v)$. Die Dimension m von $\tau(u, v)$ heißt der Charakter von L ($3\leq m\leq 5$). Der Ort aller Punkte $\lambda x+\mu y$ auf den Geraden von L ist eine dreidimensionale Mannigfaltigkeit $V_3(L)$ und $\tau(u, v)$ ist der kleinste lineare Raum, der die Tangentialräume an $V_3(L)$ in allen Punkten der Geraden $p(u, v)$ enthält. — Neben L betrachtet Verfasser eine zweite Kongruenz L', bezogen auf die nämlichen Parameter u, v. Dann bestimmen diese gemeinsamen Parameter u, v zwischen den Geraden von L und L' die Korrespondenz $\mathbf{T}$. Die Korrespondenz $\mathbf{T}$ heißt eine Punktdeformation von L in L', wenn mit $\mathbf{T}$ eine gewisse Punkttransformation $\mathbf{T}^*$ zwischen $V_3(L)$ und $V_3(L')$ einhergeht, derart daß für jedes (u, v) eine Homographie $\mathbf{K}(u, v)$ zwischen $\tau(n, v)$ und $\tau'(u, v)$ existiert von der besonderen Art, daß für jeden Punkt z der Geraden $p(u, v)$ derjenige Teil von $\mathbf{K}$, der den Tangentialraum von $V_3(L)$ im Punkt z betrifft, Tangentialhomographie der Korrespondenz $\mathbf{T}^*$ in Bezug auf den betrachteten Punkt z ist. Eine solche Homographie kann nur durch Projektivitäten $\pi(u, v)$ zwischen den Punkten von $p(u, v)$ und denen von $p'(u, v)$ vermittelt werden. Diese Projektivitäten realisieren die Deformation. Die Homographie $\mathbf{K}(u, v)$ zwischen $\tau(u, v)$ und $\tau'(u, v)$ heißt die Tangentialhomographie der Deformation. — Zur Lösung des Problems unterscheidet Verfasser zehn verschiedene Kongruenzklassen. Eine Deformation ist nur möglich, wenn L und L' zur selben Klasse gehören. Dann bleiben die Kongruenzen innerhalb des gleichen Typus mit Ausnahme des letzten willkürlich. Nicht willkürlich bleiben die Korrespondenzen $\mathbf{T}$. Spezialfälle linearer $V_3(L)$ und $V_3(L')$ sind schon früher behandelt worden[vgl. Čech, dasselbe J. **2(77)** (1952), 167–188; MR **16**, 71; Muracchini, Boll. Un. Mat. Ital. (3) **8** (1953), 390–398; MR **15**, 742]. *M. Pinl.*

Marcus, F. Sur les invariants de la théorie projective différentielle des congruences de droits de G. Fubini. Gaz. Mat. Fiz. Ser. A. **7** (1955), 409–416. (Romanian. Russian and French summaries)

Dans cette note l'auteur établit un système de formules d'où il déduit certaines propositions de la théorie projective différentielle des congruences de droites. Il montre que, pour deux congruences de droites ayant une même nappe focale, engendrées par les tangentes d'un réseau doublement conjugué, le rapport des invariants ponctuels de Fubini est égal à l'invariant absolu de l'équation ponctuelle de Laplace du réseau. Il établit que la condition d'applicabilité projective de deux congruences de droites peut s'exprimer au moyen des invariants de Fubini. Puis il démontre que les conditions d'applicabilité projective de 1ère et de 2ème espèce de Terracini coïncident avec deux de ses formules du début.
P. Vincensini (Marseille).

Grasso, Pietro. Un teorema sulle congruenze normali in un iperspazio euclideo. Matematiche, Catania **10** (1955), 26–29.

Pan, T. K. A generalized theorem of center of geodesic curvature. Amer. Math. Monthly **62** (1955), 717–718.

Let C be a curve on a surface S in a three-dimensional Euclidean space, R the ruled surface formed by the tangents to S which are perpendicular to C at all its points, and A the point of the generator PG of R at which the plane tangent to R is perpendicular to the plane tangent to S at P. It is then a well-known theorem that A is the centre of geodesic curvature of C at P, and when R is developable, A is the point of contact of PG with the edge of regression.

In this note the author replaces the unit tangent vector t of the curve C by an arbitrary unit vector field v. In terms of the angular spread vector of v with respect to C, he defines the centre of associate curvature of v with respect to C. He then states and proves a generalization of the above theorem. *E. T. Davies* (Southampton).

Godeaux, Lucien. Sur les surfaces dont les réglées gauches asymptotiques appartiennent à des complexes linéaires. Abh. Math. Sem. Univ. Hamburg **20** (1955), 57–63.

L'auteur revient ici sur une famille de surfaces sur lesquelles, à diverses reprises, il a attiré l'attention et souligné l'intéret [voir spécialement, La théorie des surfaces et l'espace réglé, Hermann, Paris, 1934; Colloque de géométrie différentielle, Louvain, 1951, Thone, Liège, 1951, pp. 191–203; Univ. e Politec. Torino. Rend. Sem. Mat. **13** (1953–54), 39–46; MR **13**, 774; **16**, 855]. Il s'agit des surfaces dont les réglées gauches asymptotiques de l'un des deux systèmes appartiennent à des complexes linéaires. Considérant les congruences W (qui existent effectivement) dont les asymptotiques u sur une nappe focale appartiennent à des complexes linéaires, et dont les réglées gauches asymptotiques u de l'autre nappe appartiennent aussi à des complexes linéaires, l'auteur indique un moyen de déterminer les complexes linéaires

contenant ces dernières surfaces réglées, et donne des applications au cas où l'on suppose donnée l'une des nappes focales de la congruence W. *P. Vincensini.*

Upadhyay, M. D. Families of ruled surfaces. Acad. Roy. Belg. Bull. Cl. Sci. (5) 41 (1955), 1113–1129.

L'auteur revient sur la considération de trois formes différentielles quadratiques:

$$f=G_{\alpha\beta}du^\alpha du^\beta,\ \theta=\mu_{\alpha\beta}du^\alpha du^\beta,\ \phi=\xi_{\alpha\beta}du^\alpha du^\beta,$$

de la théorie des congruences rectilignes. Ces formes ont fait l'objet de travaux antérieurs de Ram Behari et R. S. Mishra, et l'application répetée de l'opération jacobienne leur associé dix familles distinctes de surfaces réglées incluses dans la congruence rectiligne envisagée. L'article actuel étudie les intersections de ces diverses surfaces réglées avec la surface de référence de la congruence, et expose les principaux résultats géométriques auxquels cette étude conduit, soit dans le cas général, soit dans quelques cas particuliers. *P. Vincensini* (Marseille).

Saban, G. Fonctions duales implicites et résultats géométriques qui en découlent. Rev. Fac. Sci. Univ. Istanbul. Sér. A. 19 (1954), 148–153 (1955). (Turkish summary)

L'auteur revient sur certains théorèmes de géométrie réglée, établis par M. Kula [même Rev. 17 (1952), 322–343; MR 14, 792] par l'emploi des vecteurs duaux et de certaines fonctions duales implicites. Les résultats du travail cité avaient besoin d'être complétes à la faveur d'une analyse plus poussée des propriétés des fonctions duales utilisées, et c'est ce qu'a fait l'auteur de la note actuelle. Parmi les résultats obtenus, nous citerons le suivant relatif aux familles de congruences synectiques: L'enveloppe d'une famille de telles congruences dépendant d'un paramètre réel est soit une surface réglée soit une congruence cylindrique, le premier cas se présentant lorsque les congruences de la famille envisagée ont des représentations sphériques distinctes, le deuxième lorsque les représentations sphériques sont confondues suivant une même courbe. *P. Vincensini* (Marseille).

Mathéev, A. Analogue à un théorème de P. Serret dans l'espace elliptique. Ann. Univ. Sofia Fac. Sci. Phys. Math. Livre 1. 48 (1953–54), 23–25. (Bulgarian. French summary)

Mathéev, A. Extension des théorèmes de Schell et Mannheim pour deux surfaces réglées associées de l'espace elliptique. Ann. Univ. Sofia Fac. Sci. Phys. Math. Livre 1. 48 (1953/54), 17–21 (1954). (Bulgarian. French summary)

Kawaguchi, Akitsugu. General direction transformation and generalized homogeneous function. Tensor (N.S.) 5 (1955), 68–70.

L'A. considera i vettori controvarianti in uno spazio ad n dimensioni [la natura dello spazio non essendo precisata], e una trasformazione della forma: $'v^a=f^a(v^i)$ $(a, i=1, \cdots, n)$; diremo che tale trasformazione è una trasformazione (generale) conservante le direzioni ("general direction transformation"), se ogni coppia di vettori aventi la stessa direzione è trasformata in una coppia avente la stessa proprietà, cioè se $f^a(\varrho v^i)=\sigma(\varrho, v^i)f^a(v^i)$ per ogni scelta dello scalare ϱ $(\neq 0)$. Si studia la natura della funzione σ, sotto ipotesi di derivabilità per le f^a, e si dànno alcuni esempi. *V. Dalla Volta* (Roma).

Akbar-Zadeh, Hassan. Sur les isométries infinitésimales d'une variété finslérienne. C. R. Acad. Sci. Paris 242 (1956), 608–610.

The author in this note gives necessary and sufficient conditions for a one-parameter group of transformations to give rise to isometric motions of a Finsler space. The conditions are in terms of various tensors associated with the metric and are too technical to reproduce here. *L. Auslander* (Princeton, N.J.).

Laugwitz, Detlef. Zur projektiven und konformen Geometrie der Finsler-Räume. Arch. Math. 7 (1956), 74–77.

This paper contains two results. The first characterizes the twice differentiable conformal mappings of a Minkowski space as isometric transformations followed by homothetic expansions. The second theorem states the following: Let us consider two Finsler metrics

$$\bar g_{ij}(x; x')dx^i\, dx^j \text{ and } g_{ij}(x; x')dx^i\, dx^j$$

which are both projectively and conformally equivalent. Then there exists a constant ϱ^2 such that $\bar g_{ij}(x; x')=\varrho^2 g_{ij}(x; x')$ for all x and x'. *L. Auslander.*

Varga, O. Die Krümmung der Eichfläche des Minkowskischen Raumes und die geometrische Deutung des einen Krümmungstensors des Finslerschen Raumes. Abh. Math. Sem. Univ. Hamburg 20 (1955), 41–51.

Let $L(x)$ be the distance function of a convex surface in n-space; that is $L(x)$ is positively homogeneous of degree one, $L(x)\geqq 0$ and $L(0)=0$. The hypersurfaces $L^2(x)=1/K$ $(K>0)$ are the analogs of concentric spheres in Euclidean space. Let them have parametric equations: $x^i=x^i(u^1, \cdots, u^{n-1})$.

If v^i is an arbitrary direction, let $g_{ik}(v)=\frac{1}{2}\partial^2L^2(v)/\partial v^i\partial v^k$. When v^i is a transversal to one of the hypersurfaces, v^i are functions $v^i[x(u)]$. Then the author defines a Riemannian metric on the hypersurface by: $\gamma_{\alpha\beta}(u)=g_{ik}[x(u)]\dfrac{\partial x^i}{\partial u^\alpha}\dfrac{\partial x^k}{\partial u^\beta}$. In the Euclidean case this metric is of constant curvature. In the present case this curvature is constant if and only if the curvature of the n-space is constant. This space curvature is defined in terms of the curvature S_{hkij} of a Finsler space, and thus becomes a geometric interpretation of this tensor. *C. B. Allendoerfer.*

See also: Shiffman, p. 632; Rizza, p. 662.

Riemannian Geometry, Connections

Castoldi, Luigi. Significato geometrico del divario Riemanniano nelle connessioni metriche in X_n. Rend. Sem. Fac. Sci. Univ. Cagliari 25 (1955), 15–20.

Following Bompiani's geometrical interpretation of the torsion tensor of a general linear connexion [Boll. Un. Mat. Ital. 6 (1951), 273–276; MR 13, 688], the author examines the divergence associated with metric connexions, in particular those associated with reduced metrics introduced in an earlier paper [Atti. Accad. Ligure Lett. 9 (1952), 5–14; MR 15, 169]. *T. J. Willmore* (Liverpool).

Sasaki, Shigeo; and Goto, Morikuni. Some theorems on holonomy groups of Riemannian manifolds. Trans. Amer. Math. Soc. 80 (1955), 148–158.

I principali risultati del lavoro sono i seguenti. Data una varietà riemanniana V, siano H, H^0, h, h^0 rispettiva-

mente il gruppo di olonomia, il gruppo di olonomia ristretto, e gli analoghi gruppi omogenei. Teor. 1: Supponiamo che h^0 sia riducibile, e siano $E_{(1)}, \cdots, E_{(m)}$ i sottospazio [dello spazio tangente a V nel punto base per i gruppi considerati] irriducibili invarianti. Se $E_{(1)}$ non è invariante per h, e $\dim E_{(1)} \geq 2$, consideriamo lo spazio irriducibile invariante per h e contenente $E_{(1)}$ e denotiamolo con $E_{(1)}^*$. È allora possibile scegliere, da $E_{(1)}, \cdots, E_{(m)}$ l_1 ($l_1 = \dim E_{(1)}^* / \dim E_{(1)}$) spazi lineari della stessa dimensione, i quali individuano $E_{(1)}^*$, è ognuno dei quali è trasformato in un altro di essi da qualche trasformazione di h. Teor. 3: Sia M_n una varietà riemanniana completa il cui gruppo di olonomia h^0 si decompone, lasciando invarianti r vettori, che individuino un r-piano invariante per h, ed M_n soddisfi inoltre all'ipotesi W [v. più sotto]. Allora la parte ad r dimensioni corrispondente allo r-piano invariante del gruppo $H(M_n)$ è un gruppo discreto di movimenti privo di punti fissi in E_n. [Per l'ipotesi W cf., A. G. Walker, Proc. London Math. Soc. (3) 3 (1953), 1–19; MR 15, 159.] Teor. 5: Se il gruppo H^0 di una varietà riemanniana completa M_n lascia fisso un punto, M_n è una „Raumform" euclidea. *V. Dalla Volta.*

Lichnerowicz, André. Transformation infinitésimales conformes de certaines variétés riemanniennes compactes. C.R. Acad. Sci. Paris **241** (1955), 726–729.

L'auteur considère une variété riemannienne V_m compacte et orientable et étudie des transformations infinitésimales conformes dans V_m. Il démontre tout d'abord le théorème suivant. Pour qu'une 1-forme ξ définisse sur une variété riemannienne compacte et orientable une transformation infinitésimale conforme, il faut et il suffit qu'elle satisfasse à

$$\Delta\xi + \left(1 - \frac{2}{m}\right) d\delta\xi = K\xi,$$

où Δ est le laplacien de G. de Rham, d et δ les opérateurs de différentiation et codifférentiation et K un opérateur sur les 1-formes α defini par $K: \alpha_i \to 2R_{ij}\alpha^j$. R_{ij} étant le tenseur de courbure de Ricci.

De ce théorème, on déduit aisément le théorème suivant: Pour que ξ définisse une isométrie infinitésimale, il faut et il suffit que

$$\Delta\xi = K\xi, \quad \delta\xi = 0$$

[Yano et Bochner, Curvature and Betti numbers, Princeton, 1953; MR 15, 989].

Il considère ensuite un espace d'Einstein: $R_{ij} = \frac{1}{2}\lambda_0 g_{ij}$ ($\lambda_0 > 0$; $\lambda_1 = \frac{1}{2}\lambda_0 m(m-1)^{-1}$) et il démontre: Dans un espace d'Einstein compact et orientable, les valeurs propres de l'operateur Δ sur les 1-formes cofermées ou fermées sont supérieures ou égales respectivement aux nombres λ_0 et λ_1. Les 1-formes propres cofermées correspondant à λ_0 engendrent la sous-algèbre L_1 des isométries infinitésimales. Les 1-formes propres fermées correspondant à λ_1 engendrent un espace vectoriel L_2 des transformations infinitésimales conformes. L'algèbre L des transformations infinitésimales conformes est la somme directe

$$L = L_1 + L_2$$

avec

$$[L_1, L_2] \subset L_2, \quad [L_2, L_2] \subset L_1.$$

Enfin il suppose que notre espace d'Einstein soit un espace pseudo-kählerien et il démontre: Sur une variété irréductible pseudo-kählerienne espace d'Einstein E_{2n} ($n > 1$), toute transformation infinitésimale conforme est un automorphisme infinitésimal. Le plus grand groupe connexe de transformations conformes de E_{2n} se compose d'automorphismes. *K. Yano* (Tokyo).

Vranceanu, Georges. Sur les espaces à connexion projective. C. R. Acad. Sci. Paris **242** (1956), 61–63.

Given a space with projective connection $P_n(x^1, x^2, \cdots, x^n)$ whose coefficients are $\Gamma_{jk}{}^i, \Gamma_{jk}{}^0$. The system of partial differential equations:

$$(1) \qquad \frac{\partial^2 z}{\partial x^j \partial x^k} + \Gamma_{jk}{}^i \frac{\partial z}{\partial x^i} + \Gamma_{jk}{}^0 z = 0$$

can then be associated with P_n because under the transformations

$$z' = ze^{-\alpha}, \quad x'^i = x'^i(x', x^2, \cdots, x^n), \quad \alpha = \alpha(x', x^2, \cdots, x^n),$$

the Γ's in (1) change according to the same formulas as in P_n. It follows that all the properties of P_n are concentrated in (1).

The case when (1) is completely integrable has already been treated by A. Urban [Nederl. Akad. Wetensch. Proc. **52** (1949), 855–867; MR 11, 617]. The present author considers also the case when (1) is not completely integrable, but admits m independent solutions ($1 \leq m < n$). It is shown that if $m = 1$, P_n is a space with affine connection, and if $m > 1$, P_n is a subprojective space (in the sense of V. Kagan), $(m-1)$times projective. *R. Blum* (Saskatoon, Sask.).

Castoldi, Luigi. Caratterizzazione a priori delle connessioni metriche in X_n. Rend. Sem. Fac. Sci. Univ. Cagliari **25** (1955), 21–25.

Conditions are given which determine whether an affine connexion is a metric connexion. It is shown that the condition $I_{ij(k}{}^l a_{k)l} = 0$ is not sufficient. *T. J. Willmore* (Liverpool).

Rapcsák, A. Theorie der Bahnen in Linienelementmannigfaltigkeiten und eine Verallgemeinerung ihrer affinen Theorie. Acta Sci. Math. Szeged **16** (1955), 251–265.

This paper is concerned with a generalization of the geometry of paths as developed by Douglas [Ann. of Math. (2) **29** (1928), 143–168]. Whereas Douglas considers a system of paths represented by $x^i = f^i(t, a)$ and studies their invariants under transformations of the form

(a) $\bar{x}^i = \bar{x}^i(x)$, (b) $t = \varphi(\tau, a)$, (c) $\bar{a} = \bar{a}(a)$.

The present author is concerned with a space F_{2n-1} of line-elements (x^i, v^i) so that the paths are represented by the two sets of equations $x^i = f^i(t, a)$, $v^i = \varphi^i(t, a)$ and the transformation (a) above becomes a combination of $\bar{x}^i = \bar{x}^i(x)$ and $\bar{v}^i = v^k \partial\bar{x}^i / \partial x^k$. Correspondingly the differential equations of the paths become generalized from the set $d^2x^i/dt^2 = G^i(x, p)$, with $p^i = dx^i/dt$, to the sets $dv^i/dt = H^i(x, p, v)$; $d^2x^i/dt^2 = G^i(x, p, v)$, where H^i are homogeneous functions of the first degree in p and v, and where G^i are homogeneous of the second degree in p and of degree zero in v.

Affine connection in the space is defined by means of 3 sets of connection parameters which depend on x, p and v, with corresponding fundamental tensors of curvature and torsion. The equivalence problem is then treated and a theorem stating that a complete system of differential invariants may be formed from the fundamental tensors. *E. T. Davies* (Southampton).

Haimovici, Adolf. Sur quelques invariants dans les espaces à connexion affine. Acad. R. P. Romîne. Bul. Şti. Secţ. Şti. Mat. Fiz. 7 (1955), 595–622. (Romanian. Russian and French summaries)

In two previous papers [Com. Acad. R. P. Romåne 1 (1951), 157–163; Acad. R. P. Române. Fil. Iaşo. Stud. Acad. Cerc. Şti. 2 (1951), 66–82; MR 17, 408] the author has studied spaces with affine connection for which a function f involving a point x^i and two directions X^i, Y^i remains invariant under parallel transport. In particular the case of two dimensions has been treated. In the present paper the author considers the case of three dimensions.

The problem leads to the consideration of a system of linear partial differential equations S whose coefficients are linear in X^i and Y^i. Because f depends upon the nine variables (x^i, X^i, Y^i) it is concluded that S will have to contain at most eight linearly independent equations in order to admit one invariant, at most seven in order to admit two invariants and so on. The author limits himself to the case when the linear relations between the left-hand sides of the equations of S are independent of (X^i, Y^i).

There follows an algebraic study of a certain subsystem of S which leads, if the existence of only one invariant is assumed, to 8 different cases. For each of these cases S is integrated and the fundamental invariant given in explicit form. The same is done if the existence of two, three or four invariants is assumed; this results in five, three or one cases respectively. Finally the author shows that these spaces admit in general a field of parallel directions and that they include as a particular case the Riemannian spaces.

R. Blum (Saskatoon, Sask.).

Gołab, S.; und Kucharzewski, M. Zur Theorie der geometrischen Objekte. Ann. Polon. Math. 2 (1955), 250–253 (1956).

The author considers certain algebraic combinations of geometric objects and determines whether these combinations are themselves geometric objects. Let $\Gamma_{\mu\lambda}{}^{\nu}$ be an affine connection, $\Gamma_\mu=\Gamma_{\mu\lambda}{}^{\lambda}$, and $A_\nu{}^{\nu'}=\partial\xi^{\nu'}/\partial\xi^{\nu}$. If $T^{\lambda_1\cdots\lambda_p}_{\mu_1\cdots\mu_q}$ is an affinor-density, the combination $T^{\lambda_1\cdots\lambda_p}_{\mu_1\cdots\mu_q}\Gamma_\nu$ is not necessarily a geometric object. However, if $p=q$ and $T^{\lambda_1\cdots\lambda_p}_{\mu_1\cdots\mu_p}=A^{\lambda_1}_{\mu_1}\cdots A^{\lambda_p}_{\mu_p}$, the above combination is a geometric object. When $p=1$, we obtain $A_\mu{}^{\lambda}\Gamma_\nu=\Lambda_{\nu\mu}{}^{\lambda}$, which conceivably can be used to define the parallel displacement of a vector. This works for $n>1$ if and only if the underlying group of coordinate transformations is restricted to the affine subgroup. For $n=1$, $\Delta_{\nu\mu}{}^{\lambda}$ defines parallel displacement without this restriction.

C. B. Allendoerfer (Seattle, Wash.).

Nijenhuis, Albert. Jacobi-type identities for bilinear differential concomitants of certain tensor fields. I, II. Nederl. Akad. Wetensch. Proc. Ser. A. 58=Indag. Math. 17 (1955), 390–397, 398–403.

In this paper the author studies differential concomitants of tensor fields which do not involve linear connections and shows some applications of a concomitant which was found by the author.

Let P^{p+1} and Q^{q+1} be two contravariant tensor fields with components $P^{\kappa_0\kappa_1\cdots\kappa_p}$ and $Q^{\kappa_0\kappa_1\cdots\kappa_q}$ respectively. Schouten [Convegno Internazionale di Geometria Differenziale, Italia, 1953, Edizioni Cremonese, Roma, 1954, pp. 1–7; MR 16, 174] proved that there is a concomitant $[P, Q]^{p+q+1}$ with components

$$[P, Q]^{\kappa_0\kappa_1\cdots\kappa_{p+q}}=(\textstyle\sum_{i=0}^{p} P^{(\kappa_0\cdots\kappa_{i-1}|\mu|\kappa_i\cdots\kappa_{p-1}}\partial_\mu Q^{\kappa_p\cdots\kappa_{p+q})}+$$
$$(\textstyle\sum_{i=0}^{p}(-1)^i P^{[\kappa_0\cdots\kappa_{i-1}|\mu|\kappa_i\cdots\kappa_{p-1}}\partial_\mu Q^{\kappa_p\cdots\kappa_{p+q}]}-$$
$$(\textstyle\sum_{j=0}^{q} Q^{(\kappa_0\cdots\kappa_{j-1}|\mu|\kappa_j\cdots\kappa_{q-1}}\partial_\mu P^{\kappa_q\cdots\kappa_{p+q})}-$$
$$(\textstyle\sum_{j=0}^{q}(-1)^{pq+p+q+j} Q^{[\kappa_0\cdots\kappa_{j-1}|\mu|\kappa_j\cdots\kappa_{q-1}}\partial_\mu P^{\kappa_q\cdots\kappa_{p+q}]}.$$

For this concomitant of Schouten's, the author proves following two theorems. If $P^{(p+1)}$ is the symmetric part of P^{p+1}, and $P^{[p+1]}$ the skew symmetric part, and if $Q^{(q+1)}$, $Q^{[q+1]}$ have similar meanings, then

$$[P^{p+1}, Q^{q+1}]=[P^{(p+1)}, Q^{(q+1)}]+[P^{[p+1]}, Q^{[q+1]}]$$

and the terms on the right are symmetric and skew-symmetric respectively. For symmetric contravariant tensors P, Q, R we have

$$[P, [Q, R]]+[R, [P, Q]]+[Q, [R, P]]=0$$

and if P, Q, R are skew symmetric, we have

$$(-1)^{r(p+1)}[P, [Q, R]]+(-1)^{q(r+1)}[R, [P, Q]]+$$
$$(-1)^{p(q+1)}[Q, [R, P]]=0.$$

The author next discusses a concomitant found by himself. Let $h^{\kappa}_{\cdot\lambda}$ and $k^{\kappa}_{\cdot\lambda}$ be two tensor fields; then the concomitant is given by

$$[h, k]^{\kappa}_{\cdot\mu\lambda}=h^{\varrho}_{\cdot[\mu}\partial_{|\varrho|}k^{\kappa}_{\cdot\lambda]}+k^{\varrho}_{\cdot[\mu}\partial_{|\varrho|}h^{\kappa}_{\cdot\lambda]}-h^{\kappa}_{\cdot\varrho}\partial_{[\mu}k^{\varrho}_{\cdot\lambda]}-k^{\kappa}_{\cdot\varrho}\partial_{[\mu}h^{\varrho}_{\cdot\lambda]}.$$

For $h=k$ the concomitant $H=[h, h]$ is of special importance. The author states following four applications of H. (i) Let h have distinct real eigenvalues at every point of a manifold M over which the field is defined. The condition that the p-planes spanned by any p eigenvector fields be at each point tangent to a family of p-dimensional submanifolds X_p is that H should satisfy an identity of the form

$$H^{\kappa}_{\cdot\mu\lambda}=\underset{0}{p}_{[\mu}A^{\kappa}_{\lambda]}+\underset{0}{p}_{[\mu}h^{\kappa}_{\cdot\lambda]}+\cdots+\underset{n-2}{p}_{[\mu}\overset{n-2}{h}{}^{\kappa}_{\cdot\lambda]},$$

where the covariant vectors $\underset{0}{p}, \cdots, \underset{n-2}{p}$ are suitably chosen and h^q is the qth power of h. (ii) Let h satisfy $h^{\kappa}_{\cdot\lambda}h^{\lambda}_{\cdot\mu}=-A^{\kappa}_{\mu}$; then h is said to determine an almost complex structure. Eckmann and Frölicher's necessary condition that the structure be a complex structure is equivalent to the vanishing of H. (iii) If h is idempotent, that is, $h^{\kappa}_{\cdot\lambda}=h^{\kappa}_{\cdot\nu}h^{\nu}_{\cdot\lambda}$, the tangent space T splits up into an invariant subspace V, consisting of all $v^{\kappa}=h^{\kappa}_{\cdot\lambda}u^{\lambda}$, $u\in T$, and a null-space W consisting of all v^{κ} such that $h^{\kappa}_{\cdot\lambda}v^{\lambda}=0$. Here V and W intersect in the null-vector only. The author proves: The torsion 2-form $H(u, v)$ with values in T can be written as the sum of two terms, the first of which depends on the V-parts $h^{\kappa}_{\cdot\lambda}u^{\lambda}$, $h^{\kappa}_{\cdot\lambda}v^{\lambda}$ of u, v only and has values in W, while the second term depends on the W-parts $h'^{\kappa}_{\cdot\lambda}u^{\lambda}$, $h'^{\kappa}_{\cdot\lambda}v^{\lambda}$ only and has values in V, where $h'^{\kappa}_{\cdot\lambda}=A^{\kappa}_{\lambda}-h^{\kappa}_{\cdot\lambda}$. (iv) If h is idempotent and the spaces W are integrable, we can construct a fibre bundle and develop a theory of holonomy.

In the last section, the author gives a new concomitant of two skew symmetric forms $M^{\kappa}_{[q]}$, $N^{\kappa}_{[r]}$ of degrees q and r respectively with values in the tangent bundle:

$$[M, N]^{\kappa}_{\cdot\mu_1\cdots\mu_q\nu_1\cdots\nu_r}=M^{\tau}_{\cdot[\mu_1\cdots\mu_q}\partial_{|\tau|}N^{\kappa}_{\cdot\nu_1\cdots\nu_r]}-N^{\tau}_{\cdot[\nu_1\cdots\nu_r}\partial_{|\tau|}M^{\kappa}_{\cdot\mu_1\cdots\mu_q]}-$$
$$qM^{\kappa}_{\cdot\tau[\mu_2\cdots\mu_q}\partial_{\mu_1}N^{\tau}_{\cdot\nu_1\cdots\nu_r]}+rN^{\kappa}_{\cdot\tau[\nu_2\cdots\nu_r}\partial_{\nu_1}M^{\tau}_{\cdot\mu_1\cdots\mu_q]},$$

which is a generalisation of $[h, k]$.

K. Yano (Tokyo).

Couty, Raymond. Sur une inégalité relative aux espaces kählériens harmoniques. C. R. Acad. Sci. Paris **242** (1956), 65–67.

The author obtains an inequality satisfied by the derivatives $f'(0)$ and $f''(0)$ of the function $f(\Omega)$ associated with a harmonic Kählerian space. Equality holds only when the space has constant holomorphic curvature. This inequality is very similar to one obtained previously by Lichnerowicz for a general harmonic space [Bull. Soc. Math. France **72** (1944), 146–168; MR **7**, 80].

T. J. Willmore (Liverpool).

See also: Fichera, p. 604; Santaló, p. 656; Finzi, p. 675; Castoldi, p. 675; Brinis, p. 676; Lynn, p. 691.

Complex Manifolds

Rizza, Giovanni Battista. Dirichlet problem for n-harmonic functions and related geometrical properties. Math. Ann. **130** (1955), 202–218.

Let D be a bounded domain in the space of n complex variables, let the boundary $\Phi=\{\varphi=0\}$ be connected, analytic, and have only simple points. Given a function $u(P)$ on Φ which is the restriction of the real part $U(P)$ of a function holomorphic in D. Then $U(P)$ can be expressed in D by the integral formula

$$U(P)=\frac{(\eta-2)!}{4\pi^n}\int_\Phi\left[u(Q)\frac{\partial}{\partial n}\frac{1}{r^{2n-2}}+\frac{\Delta_r{}^2u}{\eta^3\mathfrak{L}(\varphi)r^{2n-2}}\right]d\Phi$$

where

$$\eta^2=\left(\sum_{\nu=1}^n\frac{\partial\varphi}{\partial z_\nu}\frac{\partial\varphi}{\partial \bar z_\nu}\right)^{-1}$$

$\mathfrak{L}(\varphi)$ is a generalization of Levi's differential expression introduced by the author, and $\Delta_r{}^2$ is a certain differential parameter of second order. The formula follows from Green's formula by expressing the normal derivative $\partial U/\partial n$ by u and φ though the "fundamental relation"

$$\Delta_r{}^2u+\eta^2\mathfrak{L}(\varphi)\frac{\partial U}{\partial n}=0.$$

The author's result is the extension of a corresponding formula by Martinelli [Atti Accad. Italia. Mem. Cl. Sci. Fis. Mat. Nat. **12** (1942), 143–167; MR **8**, 203] for the case $n=2$. The result is proved mainly by methods of differential geometry. For hyperplanoids (analytic polyhedra) one has $\mathfrak{L}(\varphi)\equiv0$ and $\Delta_r{}^2u\equiv0$. Finally, the transformation properties of $\mathfrak{L}(\varphi)$ are discussed: $\mathfrak{L}(\varphi)=\mathfrak{L}'(\varphi)|J|\cdot|\bar J|$ (where $|J|\cdot|\bar J|$ is the Jacobian) holds for $n=2$ for all pseudo-conformal transformations, for $n>2$ only for a certain subclass.

H. J. Bremermann (Princeton, N.J.).

Grauert, Hans; et Remmert, Reinhold. Fonctions plurisousharmoniques dans des espaces analytiques. Généralisation d'une théorème d'Oka. C. R. Acad. Sci. Paris **241** (1955), 1371–1373.

The notion of "plurisubharmonic function" is extended to "general analytic spaces" and the following notion defined: the closed set D in the analytic space X of dimension n is "analytically meager" of order k ($1\leq k\leq n$) if for every point $x\in D$ there exists a neighborhood U and an analytic set M in U of dimension not larger than $n-k$ such that: $D\cap U\subset M$. Similarly "analytically meager sets" on the boundary of "domaines etalés" (over the space of n complex variables C^n) are defined. The following is announced: Let D be an analytically meager set of order 1 in X. Let $p(x)$ be plurisubharmonic in $X-D$ and bounded in a neighborhood of every point $x\in D$. Then $p(x)$ can uniquely be continued as a plurisubharmonic function into X. If D is meager of order 2, the same result holds even without the assumption that $p(x)$ is bounded. [More general results for less general spaces have been announced in the paper reviewed below.] An application to pseudo-convex domains yields: If a "domaine etalé" over C^n is pseudo-convex at every boundary point with the exception of a meager set on the boundary, then it is pseudo-convex.

H. J. Bremermann (Princeton, N.J.).

Lelong, Pierre. Prolongement d'une fonction plurisousharmonique sur certains ensembles de capacité nulle. C. R. Acad. Sci. Paris **242** (1956), 55–57.

The author observes that the notions "subharmonic functions," "plurisubharmonic function", and "set of capacity zero" can immediately be extended to complex manifolds. He shows: Let D be a domain on a complex manifold, let $E\subset D$ be a locally compact set of capacity zero. Then every function that is plurisubharmonic in $D-E$ and bounded in a neighborhood of each point of E can uniquely be continued as a plurisubharmonic function into D. This result follows from the following lemma: Let $V(P)$ be a subharmonic function and let the property of being subharmonic be invariant with respect to linear coordinate transformations with non-vanishing Jacobian. Then $V(P)$ is plurisubharmonic. Also a theorem on the continuation of plurisubharmonic functions omitting the assumption that $V(P)$ is bounded and imposing stronger conditions on E is announced. (Compare the similar results by Grauert and Remmert reviewed above.)

H. J. Bremermann (Princeton, N.J.).

Koszul, Jean-Louis. Sur les groupes simplement transitifs d'automorphismes analytiques. C. R. Acad. Sci. Paris **241** (1955), 847–849.

Let V be a complex manifold admitting a simply transitive group G of analytic automorphisms. V has a volume element ω (unique up to a constant multiple) invariant under G. Associated with ω in the usual manner, there are a hermitian form $\mathcal{B}$ and a differential 2-form Ω. Moreover, an invariant Pfaffian form ψ is defined with the property that $d\psi=2\Omega$. This note is concerned with these three forms when V is a simply connected symmetric bounded domain.

H. C. Wang (Seattle, Wash.).

Yano, Kentaro. Quelques remarques sur les variétés à structure presque complexe. Bull. Soc. Math. France **83** (1955), 57–80.

This is an exhaustive and readable account of recent work being done on the local differential geometry of almost complex, complex, Hermitian, Kählerian structures and affine connections. Of historical interest is the fact that the so-called Kählerian metric was studied earlier by Schouten [Akad. Wetensch. Amsterdam. Proc. **32** (1929), 457–465] and Schouten and van Dantzig [Math. Ann. **103** (1930), 319–346], although it should be kept in mind that its importance is largely due to the global properties derived by W. V. D. Hodge in his study of harmonic integrals on algebraic varieties. Subjects treated in this paper include: 1) existence of certain affine connections on complex manifolds; 2) different formulations of the integrability conditions of an almost

complex structure and their identification; 3) Condition for an almost complex manifold with Hermitian metric to be integrable. *S. Chern* (Chicago, Ill.).

See also: Nijenhuis, p. 661; Couty, p. 662.

Algebraic Geometry

Hutcherson, W. R. Su alcune involuzioni cicliche dotate di periodo non inferiore a 157. Matematiche, Catania **10** (1955), 15–17.

Morin, Ugo. Alcuni problemi di unirazionalità. Univ. e Politec. Torino. Rend. Sem. Mat. **14** (1954–55), 39–53.
Expository paper.

Roth, Leonard. Alcuni problemi di razionalità per le varietà algebriche. Univ. e Politec. Torino. Rend. Sem. Mat. **14** (1954–55), 105–114.
Expository lecture. *J. A. Todd.*

Permutti, Rodolfo. Sulla superficie delle coppie ordinate di punti di una curva algebrica, a moduli generali. Ricerche Mat. **4** (1955), 48–57.

Permutti, Rodolfo. Sulla superficie delle coppie ordinate di punti di una curva algebrica di genere p, a moduli generali. Ricerche Mat. **4** (1955), 160–176.

Étude de la surface F, représentative des couples ordonnés de points d'une courbe C de genre p, en faisant tendre C vers une courbe rationelle C_0, en sorte que F tende vers F_0, image des couples ordonnés de C_0. La 1ère note est consacré au cas $p=1$; en ce cas C est une cubique générale, C_0 une cubique nodale; on peut alors représenter F_0 par une quadrique Q, sur laquelle un quadrilatère formé de 4 génératrices a_1, a_2, b_1, b_2 est l'image des couples formés avec le point double. A une correpondance $T_{n,n',0}$ à valence nulle sur C, correspond une courbe de Q contenant les génératrices a_i, b_i, ou coupant les a_i et les b_i en des points conjugués dans les projectivités définies dans chaque système de génératrices par les a, b. Ces courbes s'étudient facilement sur la représentation plane de Q, où elles sont des $C_{n+n'}(A^{n'}, B^n)$ coupant les images des a, b, dans les projectivités indiquées. Elles forment ainsi un système algébrique de dimension $nn'+1$ formé de ∞^2 systèmes linéaires irréductibles, ayant chacun pour dimension $nn'-1$, leur degré virtuel est $2nn'$, leur genre virtuel $nn'+1$. Dans le cas d'une correspondance $T_{n,n',g}$ à valence g, on trouve en tenant compte du passage g fois [ou $|g|$ fois si $g<0$] aux points A_{12}, A_{21} [ou A_{11}, A_{22}] $(A_{ij}=(a_i, b_j))$, un système de dimension $nn'-g^2+|g|+1$ formé de ∞^2 systèmes linéaires irréductibles de dimension $nn'-g^2+|g|-1$, dont les images ont les degré virtuel $2nn'-2g^2$ et genre virtuel $nn'-g^2-|g|+1$.

La 2ème note traite du cas général où C est une courbe de genre p représentés par une C_m dotée de d points doubles, que l'on fait tendre vers une C_0 représenté par une C_m dotée de $d+p$ points doubles. L'image de F_0 sera encore une quadrique Q, sur laquelle p quadrilatères $(a_1^i, a_2^i, b_1^i, b_2^i)$ sont images des couples contenant les points doubles. Comme précédemment, les correspondances $T_{n,n',g}$ à valence g de C ont pour limites des correspondances à valence g dans le champ neutre individualisé par les p couples de points qui tendent vers les points doubles supplémentaires de C_0. L'étude se fait encore sur la représentation plane de Q, où l'on étudie les $C_{n+n'}(A^{n'}, B^n)$ passant g fois [ou $|g|$ fois si $g<0$)] en A_{12}^i, A_{21}^i [ou A_{11}^i, A_{22}^i], coupant les génératrices a^i, b^i en couples conjugués dans les projectivités indiquées. On en déduit que si $n>n_0+2p-2$, $n'>n_0'+2p-2$ n_0 et n_0' étant les plus petites valeurs de n et n' pour lesquelles existe une courbe irréductible correspondant à une $T_{n_0,n_0',g}$, il y a un système algébrique de dimension $nn'+(1-p)n+(1-p)n'+p(p+g-g^2)$ formé d'∞^{2p} systèmes linéaires ayant chacun la dimension $nn'+(1-p)n+(1-p)n'+p(p-2-g^2+g)$ de degré virtuel $2nn'-2pg^2$ et genre virtuel $nn'+(p-1)n+(p-1)n'-pg^2-pg+1$. *B. d'Orgeval* (Dijon).

Godeaux, Lucien. Sur la surface des couples de points de la quintique de Snyder. Acad. Roy. Belg. Bull. Cl. Sci. (5) **41** (1955), 1258–1263.

Spampinato, Nicolò. Teoremi relativi alle ipersuperficie dell' S_r m-potenziale. Matematiche, Catania **10** (1955), 1–14.

Spampinato, Nicolò. Su alcune varietà dell' S_{11} determinate dall'algebra triduale come sottoalgebra dell'-algebra dei numeri doppio-biduali. Rend. Accad. Sci. Fis. Mat. Napoli (4) **21**, 10–24 (1954).

Spampinato, Nicolò. Su alcune varietà dell' S_{23} complesso determinate dall'algebra dei numeri quadriduali come sottoalgebra dei numeri triplo-biduali. Rend. Accad. Sci. Fis. Mat. Napoli (4) **21**, 57–78 (1954).

Spampinato, N. La varietà V_{22} dell' S_{39} complesso determinata da una V_3^n dell' S_4 prolungata nell' S_4 cinqueduale. Rend. Accad. Sci. Fis. Mat. Napoli (4) **21**, 181–196 (1954).

An r-fold bidual algebra has an $(r+1)$-dual subalgebra, which leads to the construction in S_R $(R=2r(r+1)-1)$ of a projective model W_{d-1} $(d-1=r^2+2r-2)$ of a given algebraic hypersurface in complex projective S_r. In these articles, this situation is studied in the cases $r=2, 3, 4$, [See Spampinato, Giorn. Mat. Battaglini (5) **2(82)** (1954), 359–377; MR **17**, 411.] *G. B. Huff* (Athens, Ga.).

Spampinato, Nicolò. Sulla V_8^1 dell' S_{11} contenente tutte le $V_6^{2(n+m)}$ rispondenti alle curve di ordine n e classe m dell' S_2. Rend. Accad. Sci. Fis. Mat. Napoli (4) **21**, 157–165 (1954).

In the first of the three papers reviewed jointly above it is shown that the totality of images V_6 of all curves c in S_2 describes a V_8^7 in complex S_{11}. In this note, the geometry of this V_8^7 is studied by means of its parametric representation. In particular, parametric equation of a V_6 corresponding to a given curve in S_2 are obtained.
G. B. Huff (Athens, Ga.).

Burniat, Pol. Surfaces algébriques à système canonique dégénéré. Acad. Roy. Belgique. Bull. Cl. Sci. (5) **40** (1954), 247–261.

Dans des travaux précédents [Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) **16** (1954), 209–214; MR **16**, 164] l'A. a donné des exemples de surfaces algébrique dont le système canonique, à une courbe fixe près, se compose de courbes de genre 3 variables dans un faisceau linéaire. En poursuivant ici des recherches, et en utilisant encore des plans quadruples convenables, il donne des exemples de surfaces dont le système canonique pur se décompose en une courbe fixe et en p_g-1 courbes de genre 2 variables dans un faisceau linéaire. Pour toute valeur de p_g, et pour $4p_g-5\leq p^{(1)}\leq 4p_g+1$, $5p_g-5\leq P_2\leq 5p_g+1$ on trouve des surfaces régulières possédant la

propriété désirée. On peut construire aussi des plans quadruples irréguliers dont le système canonique présente la même dégénérescence. *E. Togliatti* (Gênes).

Burniat, Pol. Surfaces algébriques à système canonique dégénéré. Acad. Roy. Belg. Bull. Cl. Sci. (5) **41**, 441–456 (1955).

L'A. donné déjà plusieurs exemples de surfaces algébriques à système canonique dégénéré [voir l'analyse ci-dessus]; il en donne ici des autres par la même méthode que l'on peut rappeler. Soit F^0 une surface algébrique irréductible possédant une involution J^0 d'ordre 4; on suppose que J^0 soit abélienne et soit composée avec trois involutions I_1, I_2, I_3 d'ordre 2 et deux à deux permutables. L'image de J^0 soit, en particulier, un plan F; la courbe de diramation du plan quadruple F se compose alors des trois courbes E_1, E_2, E_3 qui représentent sur F les doubles des courbes, U_1 U_2, U_3 des points unis de I_1, I_2, I_3. En choisissant convenablement les courbes E_1, E_2, E_3, l'A. donne 12 exemples de surfaces régulières dont le système canonique, à une composante fixe près, est composé avec les courbes de genre trois d'un faisceau. Pour ces 12 surfaces, le genre p_g a une valeur quelconque, le bigenre P_2 prend respectivement toutes les valeurs de $9p_g+1$ à $9p_g-10$, le genre linéaire (1) a les valeurs de $8p_g+1$ à $8p_g-10$. Pour les valeurs les plus élevées de $p^{(1)}$ on a des surfaces qui dépendent d'un nombre assez grand de modules. Enfin, deux autres choix des courbes E_1, E_2, E_3 donnent lieu à des surfaces irrégulières. *E. G. Togliatti* (Gênes).

d'Orgeval, B. A propos des surfaces du 4^e ordre possédant des points doubles inflexionnels. Publ. Sci. Univ. Alger. Sér. A. 1 (1954), 307–311 (1955).

L'equation d'une surface F du 4e ordre possédant un point double conique inflexionnel Z peut être mise sous la forme suivante: $z^2f_2(xy)+f_4(xy)=0$; le point Z est à l'infini sur l'axe Oz. En projetant F de Z sur le plan double $z=0$, la courbe de diramation se compose de la conique $f_2=0$ et de la quartique $f_4=0$. L'A. donne ici une discussion des différentes positions possibles de ces deux courbes, dans le but d'obtenir toutes les surfaces du 4e ordre possédant seulement des points doubles coniques inflexionnels. On retrouve les mêmes résultats que M. Dedò avait déjà trouvés par une autre méthode [Period. Mat. (4) **31** (1953), 104–128, 176–185, 207–228; MR **15**, 343]. *E. Togliatti* (Gênes).

d'Orgeval, B. Surfaces elliptiques avec un faisceau elliptique de courbes de genre 4. Publ. Sci. Univ. Alger. Sér. A. 1 (1954), 313–336 (1955).

Les surfaces F considérées ici possèdent un faisceau K de courbes elliptiques et un faisceau elliptique C de courbes de genre 4; le nombre d'intersections $(K, C)=n$ s'appelle le déterminant de F. Les courbes K sont les trajectoires d'un groupe continu T de transformations birationnelles de F en elle-même. Pour F on a: p_g égal au genre du faisceau K, $p_a=-1$, $p^{(1)}=1$. Il peut exister des courbes K multiples selon s_i, diviseur de n; si leur nombre est t, on a:

$$(1) \qquad 6=n(2p_g-2)+\sum_1^t \frac{n}{s_i}(s_i-1).$$

La valeur de p_g ne peut surpasser 4; en supposant successivement $p_g=0$, 1, 2, 3, 4, l'A. donne, par une discussion minutieuse, toutes les solutions possibles de l'équation diophantienne (1); on trouve en total 51 solutions. Les courbes C peuvent être, en particulier, des courbes hyperelliptiques; cette particularité peut se présenter seulement pour 28 des solutions générales. *E. Togliatti* (Gênes).

Severi, Francesco. Les formes de première et de seconde espèces attachées à une variété algébrique. C. R. Acad. Sci. Paris **242** (1956), 59–61.

This note is a partial announcement of what is obviously a major work to appear shortly in the Ann. Mat. Pura Appl. The author gives a rapid and helpful survey of the main features and development of the subject and announces a proof, simpler than the one given by Hodge and Atiyah [same C.R. **239** (1954), 1333–1335; Ann. of Math. **62** (1955), 56–91; MR **16**, 951; **17**, 533] and, apparently, not subject to their (possibly restrictive) hypotheiss on singularities, of the equivalence of the two classical definitions of integrals of the second kind. *D. B. Scott* (London).

Severi, Francesco. Les irrégularités d'une variété algébrique et leurs liens avec les formes de première espèce. C. R. Acad. Sci. Paris **242** (1956), 225–227.

Announcement of further results from the Annali paper referred to in the preceding review.

An algebraic variety M_r has r-dimensional irregularity $q_r=P_g^r-P_a^r$, where P_g^r is the geometric and P_a^r the arithmetic genus. A subvariety V_h $(2\leqq h\leqq r-1)$ of M_r is ordinary if it has the same number of independent s-fold $(1\leqq s\leqq h-1)$ differential forms of the first kind as M_r. The h-dimensional irregularity q_h of V_h is independent of V_h and is called the h-dimensional irregularity of M_r. The number of linearly independent k-fold differential forms of the first kind of M being denoted by i_k, it follows from the formula $P_a^r=i_r-i_{r-1}+\cdots+(-1)^{r-1}i_1$ [conjectured by Severi, Rend. Circ. Mat. Palermo **28** (1909), 33–87 and proved by Kodaira, Ann. of Math. (2) **59** (1954), 86–134; MR **16**, 617] that $i_k=q_k+q_{k+1}$.

If R_k is the kth Betti-number of M, and the reduced Betti-number $\bar{R}_k$ is the number of independent transcendental cycles modulo the non-transcendental cycles (over which all k-forms have zero period), then $\bar{R}_k\geqq 2i_k$. From the theorem that there exist forms of the second kind with arbitrary periods (a result proved by Picard and Lefschetz for $r=k=2$ but presumably subject to the conjecture of Hodge-Atiyah in the general case), it follows that all the irregularities $q_2, \cdots, q_k$ of a variety depend only in the reduced Betti-numbers $\bar{R}_1, \cdots, \bar{R}_{k-1}$ (so that ordinary sub-varieties are, in a sense, general) and that the number ϱ_0^k of linearly independent k-forms of the second kind (modulo improper forms without periods) is equal to $\bar{R}_k$. The difference $\varrho_0^k-2i_k$ is equal to the number of relations between the real (or imaginary) parts of the periods of a form of the first kind. *D. B. Scott.*

Severi, Francesco. Sulla teoria degl'integrali semplici di 3^a specie sopra un superficie o varietà algebrica. Rend. Mat. e Appl. (5) **14** (1955), 551–563.

This paper begins with a rapid (and valuable) summary of the main ideas involved in the relationship between logarithmic curves of a simple integral of the third kind on a surface (or higher variety) and the theory of the base. By a close study of the relations of algebraic equivalence among curves of a finite set, the author is enabled to find the number of essentially independent integrals of the third kind whose logarithmic curves are included among

the curves of a given finite set and also obtains some related results.

This paper is partly a development of techniques applied in the author's memoir, Funzioni quasi abeliani [Pont. Acad. Sci. Scripta Varia, v. 4, Vatican City, 1947; MR 9, 578], but the results here are sharper and the methods definitive.

D. B. Scott (London).

Keshava Hegde, S. V. The associated form of a variety over a field of prime characteristic p. Comment. Math. Helv. 30 (1956), 124–138.

In this paper, the author proves some basic properties of associated forms of relative varieties. First seven theorems are, more or less, of formal nature. Theorems 8-9, combined together, assert the following: Let V be an irreducible variety over a field K of prime characteristic, say p. Then, its associated form can not be written as a pth power if and only if a generic point of V over K generates a separably generated extension of K. This is a known statement, and it follows, with an additional simple argument. from Proposition 8, p. 123 of Weil's "Foundations of algebraic geometry" [Amer. Math. Soc. Colloq. Publ., v. 29, New York, 1946; MR 9, 303]. The author, unaware of this, proves the assertion by a somewaht elementary method, which is commonly used in van der Waerden's earlier papers.

J. Igusa.

Hoskin, M. A. Zero-dimensional valuation ideals associated with plane curve branches. Proc. London Math. Soc. (3) 6 (1956), 70–99.

The present paper deals with the subject specified in the title, following O. Zariski [Amer. J. Math. 60 (1938), 151–204] and considering only groundfields k which are algebraically closed and of characteristic zero. To every valuation ideal it associates two integers, called the indices of the ideal, and the behaviour of them under certain (quadratic and linear) transformations — as well as other properties — are studied. This leads to appropriate definitions for the multiplicity of, and proximity relations between, simple valuation ideals of the sequence defined in $k[x, y]$ by a given valuation, possessing properties analogous to those of the classical definitions.

Then, using Du Val's proximity matrix [ibid. 58 (1936), 285–289], a matrix is obtained which depends only on the way in which the simple ideals of the above said sequence are proximate to one another, and by means of which the factors of all the ideals of the sequence are obtained. Also the length of a valuation ideal is related to its point-basis, and the construction for the polynomial basis of the ideal is discussed. Finally, the unloading problem for planes curves is stated in a new form, and a formal solution of this problem is obtained by using the previous work.

B. Segre (Rome).

See also: Rees, p. 573; Motzkin, p. 576; Myrberg, p. 603; Guggenheimer, p. 652; Meynieux, p. 655.

NUMERICAL ANALYSIS

Schwidefsky, K. Fortschritte der numerischen Mathematik. Z. Vermessungswesen 81 (1956), 1–5.

Laville, Gaston. Calcul graphique d'un produit de composition. C. R. Acad. Sci. Paris 242 (1956), 441–443.

Quartey, James. Table of inverted matrices for the solution of quadratic regression coefficients. Statistica, Bologna 15 (1955), 491.

This little table is intended for fitting the best parabola

$$y=b_0+b_1x+b_2x^2$$

to a given set of points $(1, y_1), (2, y_2), (3, y_3), \cdots, (n, y_n)$, using the criterion of least squares.

Let

$$U=\sum_{k=1}^{n} ky_k-A_1\bar{y}, \quad V=\sum_{k=1}^{n} k^2y_k-A_2\bar{y},$$

where $n\bar{y}=\sum_{k=1}^{n} y_k$, $A_1=n(n+1)/2$, $A_2=n(2n^2+3n+1)/6$. The desired coefficients b_0, b_1, b_2 are given by

$$b_1=C_{11}U+C_{12}V,\ b_2=C_{12}U+C_{22}V,\ b_0=\bar{y}-[b_1A_1+b_2A_2]/n.$$

The table gives A_1, A_2, C_{11}, C_{12}, C_{22} as function of $n=4(1)50$. The C_{ij} are given to 10 significant decimal places. The agreement between this table and the table of Davis and Latshaw [Ann. of Math. (2) 31 (1930), 52–78, pp. 72–73] is not very good for small values of n.

D. H. Lehmer (Berkeley, Calif.).

Fürst, Dario. Nota alla tabella per il calcolo dei coefficienti di regressione quadratica. Statistica, Bologna 15 (1955), 492–495.

This note explains the use of the table of J. Quartey [see the preceding review].

D. H. Lehmer.

Černyšenko, È. A. Investigation of convergence and establishment of an estimate of the error of the method of averaging in a complete normed space. Ukrain. Mat. Ž. 6 (1954), 305–313. (Russian)

Let the (not necessarily linear) operator T transform the complete normed metric space X into itself; assume that $\tau=\sup\|Tf_2-Tf_1\|/\|f_2-f_1\|<\frac{1}{3}$. The "method of averaging," attributed to Yu. D. Sokolov [same Ž. 5 (1953), 159–170; MR 15, 476], is the following iterative method of solving the equation $(*)$ $Tf=f$: Let a linear operator S be given with $\|Sf\|<\|f\|$ (all f). Let $f_1=T\alpha_1$, where α_1 is such that $\alpha_1=S(T\alpha_1)$. For $n=2, 3, \cdots$, let $f_n=T(f_{n-1}+\alpha_n)$, where α_n is such that $\alpha_n=S[T(f_{n-1}+\alpha_n)]-Sf_{n-1}$.

The author proves that $\|f-f_n\|\to 0$, where f is the unique solution of $(*)$, and proves that

$$\|f-f_n\|\leqq(2\tau)^{n+1}(1-\tau)^{-n-1}(1+\tau)(1-\tau)^{-1}\|f_1\|.$$

He gives three numerical examples from spaces of functions on $[0, 1]$, with such norms as $\max|f(x)|$ and $\max|f^{(p)}(x)|$. In each application Sf is a function having a constant value, namely the mean value of f.

G. E. Forsythe (Los Angeles, Calif.).

Černišenko, E. A. The method of averaging applied to the determination of eigenvalues of an operator equation. Dopovidi Akad. Nauk Ukrain. RSR 1955, 217–221. (Ukrainian. Russian summary)

Let L be a completely continuous, self-adjoint, positive definite, linear operator on the space $C[0, 1]$ of continuous functions $[0, 1]$ with inner product $(\varphi, \psi)=\int_0^1\varphi(t)\psi(t)dt$. Let $\mu_1\geqq\mu_2\geqq\cdots$ be the eigenvalues of the problem $L\varphi=\mu\varphi$. The method of the preceding review is applied to approximate μ_1. In the present paper $S\varphi=(\varphi, e)$, where $e(t)\equiv 1$. Let $\nu_1=SLe$. Let $L_r{}^te$ be the value of L^te for ar-

gument τ. Define $\nu_2=\nu_2(\tau)$ as the larger zero of the determinant

$$\begin{vmatrix} SL^2e-\nu_2SLe & \nu_1SLe-\nu_2\nu_1 \\ L_\tau{}^2e-\nu_2L_\tau e & \nu_1L_\tau e \end{vmatrix}.$$

A related definition is given for $\nu_n(\tau)(n\geqq 3)$.

The following theorems are stated: If $SL^ie=L_\sigma{}^ie$ ($i=1, 2, \cdots$), then $\nu_n(\tau)\to\mu_1$ as $\tau\to\sigma$ and $n\to\infty$. If

$$\frac{SL^2e}{SLe}-\frac{L_\tau{}^2e}{L_\tau e}\geqq 0,$$

then $0<\nu_1\leqq\nu_2\leqq\mu_1\leqq\nu_1{}^*(\nu_2{}^*-\mu_2)(\nu_1{}^*-\mu_2)^{-1}$, where $\nu_2{}^*\geqq\mu_2$, $\nu_1{}^*\geqq\mu_2$, $\nu_i{}^*=SL^ie(SL^{i-1}e)^{-1}$ ($i=1, 2$). There is a numerical example with an ordinary differential operator L. [Misprint: In (5^2) instead of φ_2, read α_2.]

G. E. Forsythe (Los Angeles, Calif.).

Bauer, Friedrich L. Der Newton-Prozess als quadratisch konvergente Abkürzung des allgemeinen linearen stationären Iterationsverfahrens 1. Ordnung (Wittmeyer-Prozess). Z. Angew. Math. Mech. **35** (1955), 469–470.

If all proper values of $E-B^{-1}A$ lie within the unit circle with center at the origin, then the sequence of matrices Y_i defined by $Y_1=B^{-1}$ and

$$Y_{i+1}=Y_i+B^{-1}(E-AY_i)$$

converges to A^{-1} and the convergence is of first order, while

$$Y_{2i}=Y_i+Y_i(E-AY_i)$$

defines a subsequence with second-order convergence. More generally,

$$Y_{i+k}=Y_i+Y_k(E-AY_i).$$

If only the equations $Ay=b$ are to be solved, one has the alternative forms

$$Y_{2i}=y_i+Y_i(b-Ay_i),\ y_{2i}=y_i+(E-B^{-1})^iy_i.$$

A. S. Householder (Oak Ridge, Tenn.).

★ **Baetslé, P.-L. Sur les méthodes itératives de calcul numérique des vecteurs propres d'une matrice.** III^e Congrès National des Sciences, Bruxelles, 1950, Vol. 2, pp. 104–106. Fédération belge des Sociétés Scientifiques, Bruxelles.

The author presents an iterative method to calculate Λ (real diagonal) and S such that $S^{-1}AS=\Lambda$, where A is any given diagonalizable matrix with real eigenvalues. Suppose that a first solution is known, i.e., that one has L, K with $LK=I+T$ and $LAK=D+E$, where D is diagonal, E and T are small of first order, and all $E_{ii}=0$. Formulas are given for P, Q such that

$$(I+Q)LK(I+P)=I,\ (I+Q)LAK(I+P)=\Lambda,$$

up to second-order terms. For example, $\lambda_i=\Lambda_{ii}=D_{ii}-T_{ii}D_{ii}$. Special cases with symmetric A or unitary K are discussed, including a method of A. Magnier [C. R. Acad. Sci. Paris **226** (1948), 464–465; MR **9**, 471]. Jacobi's rotation method is represented in the same notation. The author prefers his method to Magnier's for symmetric A because K need not a priori be unitary up to second-order terms. There is a numerical example of order 5.

G. E. Forsythe (Los Angeles, Calif.).

Duleau, Jacques. Résolution numérique de certains systèmes d'équations linéaires vectorielles. C. R. Acad. Sci. Paris **242** (1956), 870–873.

Brief discussion of equations arising from mechanical structures where the matrix of coefficients includes blocks of zeros. *A. S. Householder* (Oak Ridge, Tenn.).

Riley, James D. Solving systems of linear equations with a positive definite, symmetric, but possibly ill-conditioned matrix. Math. Tables Aids Comput. **9** (1955), 96–101.

In order to facilitate the solution of the ill-conditioned system of linear equations

$$(1)\qquad Ax=b\ (A \text{ positive definite})$$

the author proposes a shift of the spectrum of A, replacing (1) by the less ill-conditioned system

$$(2)\qquad (A+kI)y=b\ (k>0).$$

From y the solution x of (1) may be computed by the Neumann expansion

$$(3)\qquad x=y+kC^{-1}y+k^2C^{-2}y+\cdots,$$

where $C=A+kI$. The computation of each term in this series involves of course the solution of a linear system of the type (2) with the same matrix but a new column of constants. An alternative is to compute exactly the residuals $r_0=b-Ax_0$ of an approximate solution x_0 of (1) and to construct a better solution x_1 by

$$z_0=C^{-1}r_0,\ x_1=x_0+z_0+kC^{-1}z_0$$

using only the two first terms of the expansion (3).

The arbitrary quantity k must be large enough to improve the condition appreciably and small enough to insure a rapid convergence of the Neumann expansion. It would be desirable to prove by numerical experiments that the difficulty of solving (1) is not replaced by the difficulty of handling the probably slowly convergent expansion (3). *E. Stiefel* (Zürich).

Madić, Petar. Sur une méthode de résolution des systèmes d'équations algébriques linéaires. C. R. Acad. Sci. Paris **242** (1956), 439–441.

Assuming the necessary inverses to exist, if

$$z=(B^1A^{-1}B-C)^{-1}(B^1A^{-1}b-c),\ y=A^{-1}(b-Bz),$$

then

$$\begin{pmatrix} A & B \\ B' & C \end{pmatrix}\begin{pmatrix} y \\ z \end{pmatrix}=\begin{pmatrix} b \\ c \end{pmatrix}.$$

The author gives a geometrical proof, in rather more complex notation, for the case B a column vector, B' a row vector, and hence C, Z and c scalars. Hence the solution of a system of linear equations is expressed in terms of the solution of two systems of order one less, with a common coefficient matrix. Hence, on forming a recursion, one solves n systems of order 1, $n-1$ systems of order 2, $n-2$ systems of order 3, $\cdots$.

A. S. Householder (Oak Ridge, Tenn.).

Clerc, D. Aspect mathématique d'un nouveau procédé mécanographique de résolution de systèmes d'équations linéaires. Rech. Aéro. no. **44** (1955), 51–54.

The Gaussian elimination process is described in terms of a characteristic computation routine corresponding to the elimination of a single variable. This step is describable in vector and matrix notation.

F. J. Murray (New York, N.Y.).

Janin, R. Résolution de systèmes d'équations algébriques linéaires d'ordre élevé, à l'aide des méthodes mécanographiques. (Emploi du calculateur électronique.) Rech. Aéro. no. **44** (1955), 47–50.

The process of Gaussian elimination based on a choice of the largest coefficient is described in matrix terms.

F. J. Murray (New York, N.Y.).

Lotkin, Mark. A set of test matrices. Math. Tables Aids Comput. **9** (1955), 153–161.

The finite Hilbert matrix $B_n=((i+k-1)^{-1})$, $i, k=1, 2, \cdots, n$, is often used for testing numerical methods in matrix computations. In order to have test data available for a non-symmetric matrix, a study and tabulations are carried out for $A_n=(sa_{ik})$ with $a_{1k}=1$, $k=1, \cdots, n$ and $a_{ik}=(i+k-1)^{-1}$ for $i=2, \cdots, n$, $k=1, \cdots, n$. It is known that the det B_n is the reciprocal of a positive integer and it is now shown that det A_n is $(-1)^{n-1}$ times the reciprocal of a positive integer. The elements of A_n^{-1} are integers, which is also known to be true for B_n^{-1}. While all the characteristic roots of B_n are positive, the tabulations indicate that for A_n, apart from the dominant characteristic root, all others have negative signs. Estimates for the characteristic roots and vectors are studied as well as the M- and P-condition numbers [see J. Todd, Proc. Cambridge Philos. Soc. **46** (1950), 116–118; MR **11**, 403]. The paper includes tables for $n=1$ to 10 of det A_n, the M- and P-condition numbers, the characteristic roots of largest and smallest absolute value and the corresponding vectors; A_6^{-1} and $H_6=(A_6)'A_6$ are also given.

O. Taussky-Todd (Washington, D.C.).

Goodey, W. J. Note on the improvement of approximate latent roots and modal columns of a symmetrical matrix. Quart. J. Mech. Appl. Math. **8** (1955), 452–453.

This is a matrix presentation of the second of the two (perturbation-type) methods suggested by H. A. Jahn [same J. **1** (1948), 131–144; MR **10**, 152]. It is similar to the matrix presentation of Jahn's first method which was given by A. R. Collar [ibid. **1** (1948), 145–148; MR **10**, 152].

John Todd (Washington, D.C.).

Peltier, Jean. Détermination de vecteurs propres de certaines matrices à déterminant faible. C. R. Acad. Sci. Paris **240** (1955), 2201–2203.

Certain matrices A have comparable diagonal elements (say between 0.5 and 1.5) which strongly dominate the off-diagonal elements. It is desired to find the eigenvector x of such an A belonging to some eigenvalue λ known approximately. The usual process is to solve a system like

$$\sum_{i=1}^{n}(a_i^j-\lambda\delta_i^j)\xi_i=0 \quad (j=1, \cdots, n-1),\ \xi_n=1,$$

(λ fixed) by elimination. This may require too much machine capacity, and the author proposes a method claimed to reduce capacity requirements. In summary, he uses a generalized Newton method to solve the system

$$(*)\qquad \begin{aligned}&\sum_{i=1}^{n}(a_i^j-\lambda\delta_i^j)\xi_i=0 \quad (j=1, \cdots, n),\\ &\sum_{i=1}^{n}\xi_i^2=1,\end{aligned}$$

and thus improve approximate values of $\xi_1, \cdots, \xi_n$, and λ. Elimination is proposed for the basic Newton step of solving the linearized form of (*).

G. E. Forsythe.

Heinrich, H. Eine Umkehrung des Hornerschen Schemas. Z. Angew. Math. Mech. **35** (1955), 468–469.

The reversal of Horner's method is described in detail, with examples. By Horner's method,

$$(1)\qquad f(x)=a_nx^n+a_{n-1}x^{n-1}+\cdots+a_1x+a_0$$

can be transformed into

$$(2)\qquad f(x)=c_0+c_1(x-x_0)+c_2(x-x_0)(x-x_1)+\cdots+c_n(x-x_0)(x-x_1)\cdots(x-x_{n-1}).$$

This note shows the method whereby (2) can easily be transformed back into (1).

E. Frank (Chicago, Ill.).

Bahvalov, N. S. Some remarks concerning numerical integration of differential equations by the method of finite differences. Dokl. Akad. Nauk SSSR (N.S.) **104** (1955), 805–808. (Russian)

The process of integration of the differential equation $y'=f(x, y)$ with the initial condition $y(x_0)=y_0$ is considered [see Bahvalov, same Dokl. (N.S.) **104** (1955), 683–686; MR **17**, 412]. It is shown that the solution of the difference equation used in the process is unstable with respect to the round-off error. An estimation of the error of the solution of the differential equation for a particular case is given.

S. Kulik (Columbia, S.C.).

Rutishauser, Heinz. Bemerkungen zur numerischen Integration gewöhnlicher Differentialgleichungen n-ter Ordnung. Z. Angew. Math. Phys. **6** (1955), 497–498.

The main result, proved elsewhere, is that in the numerical integration of ordinary differential equations, with constant interval h, the error in each step is approximately proportional to h^4 for the methods of Runge-Kutta (simultaneous first-order equations), Kutta-Nyström (second-order equation), and Kutta-Zurmühl (equation of order n). This contradicts error claim of h^{n+2} for the Kutta-Zurmühl method [R. Zurmühl, Z. Angew. Math. Mech. **28** (1948), 173–182; MR **10**, 212].

L. Fox.

Conte, S. D.; and Reeves, R. F. A Kutta third-order procedure for solving differential equations requiring minimum storage. J. Assoc. Comput. Mach. **3** (1956), 22–25.

Reed, Harry L., Jr. Numerical integration of oscillatory systems. Ballistic Research Laboratories, Aberdeen Proving Ground, Md., Rep. no. 957, 15 pp. (1955). (U.S. Government Agencies, their contractors and others cooperating in Government research may obtain reports directly from the Ballistic Research Laboratories. All others may purchase photographic copies from the Office of Technical Services, Department of Commerce, Washington 25, D. C.)

Numerical integration by machine of linear systems of differential equations with constant coefficients is considered. The Runge-Kutta and Adams methods are compared for efficiency. An example is given to illustrate numerical secularization, in which a high-frequency oscillation of the system is suppressed by numerical techniques. The only errors considered are the truncation errors introduced by the numerical methods.

W. S. Loud (Cambridge, Mass.).

Lebedev, V. I. The equations and convergence of a differential-difference method (the method of lines). Vestnik Moskov. Univ. **10** (1955), no. 10, 47–57. (Russian)

For an equation of the form

$$L_1(u)+L_2\left(\sum_1^p b_j(y)\partial^j u/\partial y^j\right)=f(x, y)$$

where $L_i=\sum_0^{m_i} A_{ki}(x)\partial^k/\partial x^k$, with appropriate boundary conditions, the author is interested in obtaining numerical solutions by representing the derivatives by differences in the y-direction only, thus replacing the partial differential equation by a system of ordinary differential equations. Most of the discussion relates to the standard parabolic and elliptic equations. *A. S. Householder.*

Nishimura, Toru. On a new method of finite differences for solving differential equations. Proceedings of the Second Japan National Congress for Applied Mechanics, 1952, pp. 303–304. Science Council of Japan, Tokyo, 1953.

The author replaces Poisson's equation $\Delta u=f(x, y)$ by the difference equation

$$(1)\qquad L[u]=6h^2f+\tfrac{1}{2}h^4(\Delta f)+\tfrac{1}{60}h^6(\Delta\Delta f+2f_{xxyy}),$$

where $L[\ ]$ is the usual nine-point "star" approximating Δ, and $h=\Delta x=\Delta y$; the derivatives of f are evaluated at the center of the star. (1) has a very small truncation error. The author gives similarly accurate difference approximants for $\Delta\Delta u=f$ and $\Delta\Delta u=k^4u$ (in the latter case, he approximates $\Delta\phi=k^2\phi$, $\Delta\psi=-k^2\psi$, where $\Delta u=k^2v$, $u=\phi+\psi$, $v=\phi-\psi$). He also discusses briefly the equations $u_t=u_{xx}$ and $u_{tt}=u_{xx}$. Finally, for the equation $u_{tt}+u_{xxxx}=0$ he proposes the difference approximant

$$u(x, t+\Delta t)=u(x, t-\Delta t)+\tfrac{1}{8}[u(x-3\Delta x, t)+u(x+3\Delta x, t)]-\tfrac{9}{8}[u(x-\Delta x, t)+u(x+\Delta x, t)],$$
$$\Delta t/\Delta x^2=\tfrac{1}{2}\sqrt{3}$$

which is a stable 6-point formula, though possessing a larger truncation error than the usual 7-point formula. [Reviewer's note: the 7-point formula is unstable for $\Delta t/\Delta x^2>\frac{1}{2}$ and hence requires at least $\sqrt{3}$ times as many steps in the t-direction as the 6-point formula proposed.] *M. A. Hyman* (Philadelphia, Pa.).

Sheldon, John W. On the numerical solution of elliptic difference equations. Math. Tables Aids Comput. **9** (1955), 101–112.

The author considers solution of the vector equation $CV=\varrho$, of the type occurring when an elliptic partial differential equation is replaced by a suitable partial difference equation. The matrix C is symmetric, positive-definite, and satisfies certain other conditions which allow iterative solution by the method of "successive over-relaxation". For a convenient ordering of the unknowns one can improve their values in the "forward" and then the "backward" direction. The author shows that a forward-and-backward sweep is generally less efficient for producing convergence than two forward sweeps. However, the eigenvalues of the forward-backward relaxation operator are all real; hence one may use Richardson's method to accelerate convergence [Philos. Trans. Roy. Soc. London. Ser. A. **210** (1910), 307–357; see also D. Young, J. Math. Phys. **32** (1954), 243–255, for discussion and further references; MR **15**, 650].

The author shows, on the basis of a reasonable assumption, that for h small (h the mesh-width for the difference equation) the number of iterations necessary with the new method to reduce the error a prescribed amount is $o(h^{\frac{1}{2}})$; for successive over-relaxation or Richardson's method alone, the number of iterations is $o(h)$. Numerical experiments with a high-speed computer support these conclusions. The author also conjectures how the speed of convergence may be further increased by using other orders of sweep and a sharper application of Richardson's technique. *M. A. Hyman* (Philadelphia, Pa.).

Saito, Shinroku. The illustrative and numerical method to get the solution of the differential equation of the conduction of heat from the aspect of the stochastic differential equation. II. Proceedings of the Fourth Japan National Congress for Applied Mechanics, 1954, pp. 349–352. Science Council of Japan, Tokyo, 1955.

The author considers solution of the generalized heat equation $c\varrho\theta_t=\lambda\theta_{xx}+\lambda_x\theta_x$, where c, ϱ, λ may be functions of x, θ. The differential equation is replaced by a difference equation in the usual way, the latter being given a stochastic interpretation. The mesh-ratio $\Delta t/\Delta x^2$ is varied in a way which allows convenient step-wise solution, numerically or graphically. When ϱ, c, λ depend only on x, standard curves may be used to obtain the solution. Several problems are worked, one having a time-variant boundary condition. Comparison with analytical results is made in two cases, and the author's technique gives close agreement. *M. A. Hyman* (Philadelphia, Pa.).

Stark, R. H. Rates of convergence in numerical solution of the diffusion equation. J. Assoc. Comput. Mach. 3 (1956), 29–40.

★**Ahlfors, L. Two numerical methods in conformal mapping.** Experiments in the computation of conformal maps, pp. 45–52. National Bureau of Standards Applied Mathematics Series, No. 42. U. S. Government Printing Office, Washington, D. C., 1955. 40 cents.

As is known, the inverse function of a function which maps a polygon conformally on the unit circle is given explicitly by the Schwarz-Christoffel formula. This formula, however, involves a certain number of unknown constants and the direct solution of the transcendental equations to which it leads is, at the present time, very laborious. The author proposes two other methods for solving the problem. Since the determination of the mapping function can be reduced to that of finding the harmonic measure of the sides of the polygon, the author is primarily concerned with the latter problem. The first method is based on the use of polynomials with certain extremal properties. Let D be a region in the z-plane bounded by a rectifiable curve C and let z_0 be a fixed interior point of D. The following minimum problem is considered: To find a polynomial $f_n(z)$ of degree n, normalized by $f_n(z_0)=1$, which makes the integral $\int_C |f_n(z)|^2|dz|$ a minimum, where the class of competing functions is the class of all polynomials of degree n, normalized as before. These polynomials converge to a function which is expressible in a simple manner in terms of the mapping function. The polynomials $f_n(z)$ are closely related to the Szegö orthogonal polynomials and are capable of representation by multiple integrals [see also, e.g., G. Pólya and G. Szegö, Aufgaben und Lehrsätze aus der Analysis, Bd. I, Springer, Berlin, 1925, pp. 48–49]. A related extremal problem for determining the harmonic

measure directly is also given. The merits of this method are discussed. The second method proposed is an extension of the alternating method of Schwarz to the case when the given polygon is replaced by more than two overlapping regions. This extension leads to the solution of an inhomogeneous integral equation to which the method of iteration is applied [for a related procedure, see R. Nevanlinna, J. Reine Angew. Math. **180** (1939), 121–128]. It is shown that the convergence is in a geometric ratio. The second method is particularly recommended for numerical work by the author. *W. Seidel.*

★**Blanch, G.; and Jackson, L. K. Computation of harmonic measure by L. Ahlfors' method.** Experiments in the computation of conformal maps, pp. 53–61. National Bureau of Standards Applied Mathematics Series, No. 42. U. S. Government Printing Office, Washington, D. C., 1955. 40 cents.

The second method proposed by Ahlfors [see the preceding review] is used to obtain the numerical solution of the Dirichlet problem for a rectangle with sides of length 4 and 8. The harmonic function is to take on the value 1 on one of the shorter sides of the rectangle and zero on the other three sides. The rectangle in question is covered by six overlapping circles, nine iterations are performed, and these are compared with the known true solution of the Dirichlet problem. Remarks are made on the computational process and on methods for speeding of convergence. Tables are appended. *W. Seidel* (Notre Dame, Ind.).

Lanczos, C. Spectroscopic eigenvalue analysis. J. Washington Acad. Sci. **45** (1955), 315–323.

Given a function $f(t)$ of the type

(1) $f(t)=a_1\cos\nu_1t+a_2\cos\nu_2t+\cdots+a_n\cos\nu_nt,\quad 0<\nu_i<\pi$

the problem is discussed of finding the frequencies ν_i from a table of $f(t)$ given in the interval $0\leqq t\leqq N$. For this purpose the Fourier transform

(2) $$F(k)=\int_0^N\cos kt\,f(t)dt$$

is used. From

$$F(k)=\frac{N}{2}\sum_{\varrho=1}^n a_\varrho\frac{\sin(k-\nu_\varrho)N}{(k-\nu_\varrho)N}+\frac{N}{2}\sum_{\varrho=1}^n a_\varrho\frac{\sin(k+\nu_\varrho)N}{(k+\nu_\varrho)N}$$

it follows that with increasing N the function $F(k)$ has in the region $k>0$ very high and steep peaks at the abscissas $k=\nu_i$ but is small elsewhere. Hence, for a suitable large N a graph of $F(k)$ permits one to locate the frequencies. A numerical technique is developed to carry out the transformation (2) if $f(t)$ is given at integer abscissas, replacing the integral (2) by a process of summation. This technique is then applied to the determination of the eigenvalues of a given n-row matrix A, assuming that A has all its eigenvalues real and contained in the interval $0<\lambda<1$. An arbitrary vector b_0 is chosen and iterated following the rule

(3) $$b_\varrho=T_\varrho(A)b_0,$$

where $T_\varrho(\lambda)$ is the Chebyshev polynomial of degree ϱ adapted to the interval (0, 1) as interval of orthogonality. Let

$$b_0=\beta_1w_1+\beta_2w_2+\cdots+\beta_nw_n$$

be in the reference-system of the eigenvectors w_i of A; then

$$b_\varrho=\beta_1w_1\cos\varrho\nu_1+\beta_2w_2\cos\varrho\nu_2+\cdots+\beta_nw_n\cos\varrho\nu_n,$$

where $\cos\nu_i=1-2\lambda_i$. Replacing the discrete variable ϱ by a continuous variable t, the first component of b_ϱ is

$$\beta_1w_{11}\cos\nu_1t+\beta_2w_{21}\cos\nu_2t+\cdots+\beta_nw_{n1}\cos\nu_nt,$$

where w_{j1} stands for the first component of the jth eigenvector. (3) is a function of the type (1). Hence the developed technique permits one to compute the frequencies ν_i and the eigenvalues λ_i. In the paper the determination of the eigenvectors is added and several refinements concerning multiple and close eigenvalues are discussed.

Because of its simple routine this promising method seems to be very well suited for automatic computation of rough estimate of all the eigenvalues of a given matrix. The method has been tested for matrices of orders 6 to 8, several hundreds of iterations of the type (3) having been performed. *E. Stiefel* (Zürich).

Nikolaev, P. V. On binary anamorphosis of N-rational equations. Dokl. Akad. Nauk SSSR (N.S.) **97** (1954), 601–604. (Russian)

Using the terminology and notation of the author's earlier papers [same Dokl. (N.S.) **88** (1953), 209–212; MR **16**, 407, and papers there cited] it is here shown that, for $n=3$, if $F(t,\tau)=0$ is nomographically rational, non-degenerate and of dimension 3 with respect to (t_3,τ_3) it admits of a simple anamorphizing factor $\psi_3(t_1,\tau_1;t_2,\tau_2)$ if and only if the rank of each of its two basic matrices $T^{(13)}$ and $T^{(23)}$ is two. This factor is unique to within a factor $\psi_1(t_1,\tau_1)\psi_2(t_2,\tau_2)$. If F is of dimension 2 with respect to (t_3,τ_3), it admits of an anamorphizing factor of the given form if and only if the auxiliary equation $F(t,\tau)+f(t,\tau)=0$ of dimension 3 does, where the complimentary function f is determined by solving a system of linear homogeneous equations. The results are applied to the cases: (a) F real, (b) the τ_i constant, (c) the equations of third to sixth nomographic order discussed in the paper referred to above. The foregoing considerations are all based on the assumption that F has been put in the nomographically rational form

$$\sum a_{ijk}f_{1i}f_{2j}f_{3k}.$$

An algorithm is given for doing this when it is possible and not evident. *R. Church* (Monterey, Calif.).

Nikolaev, P. V. Binary anamorphosis of equations admitting a simple A-factor. Dokl. Akad. Nauk SSSR (N.S.) **103** (1955), 195–198. (Russian)

Continuing the investigation in the paper reviewed above, theorems are given which permit solution of the problem of determining the existence of a simple anamorphizing factor for $F(t,\tau)=0$ in all cases, and finding the elements of the Massau determinant, using only rational operations on special values of the function F. The results are formulated in terms of two third-order square matrices. The elements of the last row of each of these are the coefficient functions in a basic representation of F in terms of a basis in (t_3,τ_3) while the other rows are these functions evaluated for particular values of their two pairs of arguments. The two cases of dimension 3 and 2 differ mainly in restrictions on these, and (in case of dimension 2) auxiliary constants. In both cases the vanishing of the determinants of both matrices (with appropriate conditions on the constants) is necessary and sufficient for existence of the simple anamorphizing factor, and the elements of the corresponding Massau determinant are the basis functions and the elements in the last row of each adjoint. The results are applied to the special case $f_3(t_3,\tau_3)=F(t_1,\tau_1;t_2,\tau_2)$. *R. Church.*

Nikolaev, P. V. On the closure of nomographic representations of equations. Dokl. Akad. Nauk SSSR (N.S.) **103** (1955), 365–368. (Russian)

The results of the paper reviewed above are used to show that if $F(t, \tau)=0$ is algebraic in t_1 then any anamorphosis will, to within equivalency, be algebraic in t_1. $F(t, \tau)$ and $G(t, \tau)$, analytic in a domain, are said to be equivalent if $F=\psi_1\psi_2\psi_3 G$, where $\psi_i=\psi(t_i, \tau_i)\neq 0$. $F=0$ and $G=0$ are equivalent if F and G are. The anamorphizing factor $\psi(t, \tau)$ here considered is

$$\psi_1(t_2, \tau_2; t_3, \tau_3)\psi_2(t_3, \tau_3; t_1, \tau_1)\psi_3(t_1, \tau_1; t_2, \tau_2)$$

so that $\Phi(t, \tau)=\psi(t, \tau)F(t, \tau)$ is equal to a Massau determinant. As a consequence, if $F(t, \tau)=0$ is algebraic in all variables the three binary fields (or scales) of any nomographic representation will consist of unicursal algebraic curves. *R. Church* (Monterey, Calif.).

Meyer zur Capellen, W. Eine interessante Kopplung zwischen Parallel-Fluchtentafel und N-Fluchtentafel. Z. Angew. Math. Mech. **35** (1955), 473–474.

See also: Albrecht, p. 628; Tamme, p. 646; Kaazik and Tamme, p. 647; Elsen and Ledoux, p. 674.

Tables

On Order

★ **Schuler, M.; und Gebelein, H. Acht- und neunstellige Tabellen zu den elliptischen Funktionen, dargestellt mittels des Jacobischen Parameters q. Eight and nine place tables of elliptical functions based on Jacobi's parameter q.** Springer-Verlag, Berlin-Göttingen-Heidelberg, 1955. xxiv+296 pp. DM 58.00.

The usual tables of elliptic functions, $\mathrm{sn}(u, k^2)$ e.g., when $0\leqq k^2\leqq 1$ and $0\leqq u\leqq K$, give values for equal intervals in k^2 [L. M. Milne-Thomson, Jacobian elliptic tables, Dover, New York, 1950; MR **13**, 987], or in $\theta=\sin^{-1} k$ [G. W. and R. M. Spenceley, Smithsonian elliptic functions tables, Washington, D.C., 1947; MR **9**, 380], and for equal intervals in u or u/K. These are not entirely satisfactory in view of the slow approach of the function to its limiting value, as $k\to 1$. Indeed the area covered by the curves $\mathrm{sn}(pK, k^2)$, $0\leqq p\leqq 1$, $0\leqq k^2\leqq .99$, appears about the same as that for the range $.99\leqq k^2\leqq 1$; in the latter range, which is of practical importance, auxiliary functions of some kind have had to be used. Schuler and Gebelein present new sets of tables which will cover a range up to $k^2=.999999$, using q as the modular parameter, and introduce new auxiliary functions. These are

$$G=G(q, z)=(-\vartheta_1(x)-2q^{\frac{1}{4}}\sin x)/2q^{9/4}\sin x=$$
$$(1+2z)+q^4(1-2z-4z^2)-q^{10}(1+4z-4z^2-8z^3)+\cdots$$

and

$$H=H(q, z)=(\vartheta_3(x)-1)/q=$$
$$2z-q^3(2-4z^2)-q^8(6z-8z^3)+\cdots.$$

Here we use the standard notations, in particular $z=\cos 2x$. These are practically linear (in q^4 and q^3 respectively) for z fixed, and quadratic in z, for q fixed. The ϑ-functions, the Jacobian elliptic functions and various elliptic integrals can be expressed in terms of G, H. For instance, with $x=\pi u/K$, we have

$$(\mathrm{sn}\, u/\sin x)=$$
$$(1+qH(1))(1-q^2G(z))/[(1-q^2G(-1))(1+qH(-z))].$$

The basic tables, which were computed by hand to 11 or 12 places, are those of G, H; they are given to 9D, for $z=-1(.05)1$ and for $q^4=0.001(.001)0.1$ and $q^3=0.002(.002)0.176$ respectively. First and second differences are given in the case of G and first differences for H. For convenience in interpolation, both tables are presented twice: first, for fixed q and varying z and then, for fixed z and varying q. The corresponding values of q, to 7D and θ in degrees and minutes, to four decimals of a minute are given.

From these tables ϑ-functions, or elliptic functions, e.g. $\mathrm{sn}\,(u, k^2)$, can be computed directly, using formulae such as that given above. Since numerical differentiation is easy in these tables the values of derivatives of the ϑ-functions, or the elliptic functions can be readily obtained. If k is given, it will be necessary first to obtain q, K and then z; a conversion table to 8D, with first differences, giving $(1-q)^{-1}$, K, K/E as functions of $-\lg k'$, for $-\lg k'=0(.005)3$ is provided for this purpose. [The values of K, E for $-\lg k'\geqq 0.5$ have been taken from E. L. Kaplan, J. Math. Phys. **25** (1946), 26–36; MR **7**, 485.]

An alternative approach to the elliptic functions is provided by a second series of tables, which give $\lg(\mathrm{sn}\, u/\sin x)$, $\lg(\mathrm{cn}\, u/\cos x)$, $\lg \mathrm{dn}\, U$ to 8D, with first differences. The evaluation of $\mathrm{sn}(u, k^2)$ will be rather easier, but will require the use of tables of antilogarithms in addition to tables of trigonometrical functions; the values obtained will be somewhat less accurate. The logarithmic tables are given, first for fixed q and varying z, then for fixed z and varying q; in each case, the range $q=0.01(.01)0.55$, $z=-1(.05)1$ is covered. Values of θ in degrees and minutes to two decimals of a minute and of $-\lg\cos\theta=-\lg k'$, K and K/E to 8D are given for each q.

The introductory material is given in German and in English. Graphs of the various functions are given. The tables are clear and appear to be reproduced from typescript, and no attempt has been made to make a confusing but compact table. Seven typical calculations are carried out in detail. These have been chosen so as to be comparable with entries in the 12D table of the Spenceley's: discrepancies are at most a couple of units in the last place. *John Todd* (Washington, D.C.).

On Order

★ **Schuler, M.; und Gebelein, H. Fünfstellige Tabellen zu den elliptischen Funktionen dargestellt mittels des Jacobischen Parameters q. Five place tables of elliptical functions bases on Jacobi's parameter q.** Springer-Verlag, Berlin-Göttingen-Heidelberg, 1955. xi+114 pp. DM 29.60.

This is a shortened version of the volume reviewed above, and is meant for the physicist, engineer, surveyor, rather than the professional computer. We use the notation of the preceding review.

The main tables are of $\lg(\mathrm{sn}\, u/\sin x)$, $\lg(\mathrm{cn}\, u/\cos x)$, $\lg \mathrm{dn}\, u$ given for $z=-1(.1)1$, and $q=0.01(.01)0.5$, to 5D with first and second differences, the latter modified where necessary. A table for fixed q, varying z is presented first, and followed by a re-arrangement in which z is fixed, q varying. The values of θ in degrees, minutes and two decimals of a minute, and of $-\lg\cos\theta=-\lg k'$, K and K/E to 5D are also given for each q.

In case the ϑ-functions, or their derivatives, are wanted, they can be obtained from tables of $\bar{G}=1-q^2G$ and $\bar{H}=1+qH$, for $q=0.01(.01)0.5$ and $z=-1(.01)1$, which are also given. These are tabulated first as functions of z, with fixed q and then as functions of q with z fixed. First and second differences are given as required.

A conversion table giving $(1-q)^{-1}$, K, K/E to 5D and θ in degrees, minutes and two decimals of a minute for $-\log k'=0(.01)2.5$ is provided. There is also a short table of Everett coefficients, and their derivatives; these are given to 4D at interval .01.

As in the larger tables, the introductory material is given in German and English, there are various graphs illustrating the behavior of the functions, and typical examples are worked in detail. The tables have been set in type and are clearly printed. *John Todd.*

★**Reynolds, George E. Table of squares of cosecants.** Electronics Research Directorate, Air Force Cambridge Research Center, Cambridge, Mass., 1954. iv+89 pp.

Cosec$^2 x$ is tabulated to eight significant figures for $x=0°(0.01)90°$, and a complementary argument facilitates extraction of sec$^2 x$. The last figure may have errors of up to one unit. Spot-checking supports this claim, but occasional misalignment of argument and respondent necessitates more than usual care in reading the table. The main part of the table was prepared from values of cosec x and sec x borrowed from the N.B.S. punched-card library. *L. Fox* (Teddington).

Rahman, A. Tables of integrals

$$A_n(\alpha)=\int_1^\infty \lambda^n e^{-\alpha\lambda}d\lambda \text{ and } F_n(\alpha)=\int_1^\infty Q_0(\lambda)\lambda^n e^{-\alpha\lambda}d\lambda.$$

Ann. Soc. Sci. Bruxelles. Sér. I. **69** (1955), 123–128.

The integrals $A_n(\alpha)$ (which is essentially an incomplete Γ-function) and $F_n(\alpha)$ are of interest in atomic physics. [$Q_0(\lambda)$ is the usual Legendre function $\frac{1}{2}\{\ln[(\lambda-1)/(\lambda+1)]\}$]. They are tabulated here for $n=0(1)10$, $\alpha=0.1(.1)3$ to 10S. Computations were carried out by recurrence relations and were to 12 figures. The formula

$$A_n(\alpha+h)=A_n(\alpha)-hA_{n+1}(\alpha)+\tfrac{1}{2}h^2A_{n+2}(\alpha)-\cdots$$

can be used for checking and interpolation; a similar formula holds for $F_n(\alpha)$. Earlier tables have been given by N. Rosen [Phys. Rev. (2) **38**(1931), 2099–2114], and M. Kotani, A. Amemiya and T. Simose [Proc. Phys.-Math. Soc. Japan (3) **20** (1938), extra no. 1; **22** (1940), extra no. 1; MR **2**, 239]. Tables for negative n are to be found in an appendix to a paper by G. Placzek [Nat. Bur. Standards Appl. Math. Ser. no. **37** (1954), pp. 57–111; MR **16**, 402]. *John Todd* (Washington, D.C.).

Lipow, M.; and Zwick, S. A. On the roots of the equation:
$$Y_1(mx)[xJ_1(x)-BJ_0(x)]-J_1(mx)[xY_1(x)-BY_0(x)]=0.$$
J. Math. Phys. **34** (1956), 308–315.

The equation

$$Y_1(mx)[xJ_1(x)-BJ_0(x)]-J_1(mx)[xY_1(x)-BY_0(x)]=0$$

occurs in several physical applications, particularly in heat conduction. Its solutions x are real, simple, and symmetrically placed about the origin for real m and B and $m\geqq 0$. The tables give all the solutions which are less than 25 for $m=0.2, 0.4, 0.6, 0.8$; $B=0.01, 0.05, 0.10, 0.20, 0.50, 1.0, 2.0, 5.0, 10, 50$ and 100. The last figure, of the six decimals given, is not expected to have an error exceeding two units. The computations were performed by inverse interpolation.

The paper also contains a discussion of methods for obtaining roots of high order and of small order, based respectively on the use of an asymptotic series expansion and an infinite product expansion. *L. Fox* (Teddington).

Zaroodny, S. J. An elementary review of the Mathieu-Hill equation of real variable based on numerical solutions. Ballistic Research Laboratories, Aberdeen Proving Ground, Md., Memo. Rep. no. **878** (1955), 29 pp. (Government Agencies, their contractors and others cooperating in Government research may obtain reports directly from the Ballistic Research Laboratories. All others may purchase photographic copies from the Office of Technical Services, Department of Commerce, Washington 25, D. C.)

A large number of solutions of Mathieu's equation have been computed on ENIAC and listed here. For the equation $y''+\varepsilon(1+k\cos t)y=0$, solutions u and v, with initial conditions $u=1$, $u'=0$, $v=0$, $v'=1$, are available to about ten decimals, and their derivatives to five decimals, for $\varepsilon=1(1)10$ and $k=0.1(0.1)1.0$. For the equation $y''+\frac{1}{4}(n^2+\gamma\cos t)y=0$ calculations have been performed for nearly 700 combinations of n^2 and γ in the range $\gamma=0$ to 16, $n^2=0$ to 12, the precise combinations chosen being marked on a chart. The behaviour of the stable solutions $e^{i\mu t}\phi(t)$, where $\phi(t)$ is periodic, and the unstable solutions $e^{\beta t}\phi(t)$, is illustrated on a second chart more extensive than any previously published, on which contours of β and of $S=e^{\mu\pi}$ are plotted with n and γ/n^2 as abscissa and ordinate respectively.

An introduction discusses methods of computation and gives an elementary introduction, for the benefit of the "occasional user of Mathieu theory", concerning the choice and behaviour, stable, unstable, and characteristic, of the various solutions of Mathieu's equation. *L. Fox* (Teddington).

Jaeger, J. C. Numerical values for the temperature in radial heat flow. J. Math. Phys. **34** (1956), 316–321.

The main table given in this paper is concerned with the temperature distribution in a region bounded internally by a circular cylinder. If the initial temperature is zero, and is raised to $f(t)$ for $t>0$, solutions can be obtained from Duhamel's theorem from the special case $f(t)\equiv 1$. For the latter, the temperature is given by

$$V=1-\frac{2}{\pi}\int_0^\infty e^{-\tau u^2}\frac{C_0(u, Ru)du}{u[J_0^2(u)+Y_0^2(u)]},$$

where τ and R are dimensionless parameters corresponding respectively to time and space, and

$$C_0(u, Ru)=J_0(u)Y_0(Ru)-Y_0(u)J_0(Ru).$$

Three-decimal values of V are given for arguments R and τ ranging respectively, at suitably varying intervals, from 1 to 100 and 0.001 to 1000. The method of calculation suggests that the error rarely exceeds half a unit in the last figure. Formulae are also given to facilitate calculation in certain ranges. *L. Fox* (Teddington).

Sternberg, R. L.; Shipman, J. S.; and Kaufman, H. Tables of Bennett functions for the two-frequency modulation product problem for the half-wave square-law rectifier. Quart. J. Mech. Appl. Math. **8** (1955), 457–467.

Tables of

$$A_{mn}^{(\alpha)}(k)=2\pi^{-2}\iint(\cos u+k\cos v)^\alpha \cos mu \cos nv\, du dv,$$

where the integral is over that part of the square $0\leqq u, v\leqq\pi$ in which $(\cos u+k\cos v)\geqq 0$ are given for $\alpha=2$, $k=0.02(.02)1$, to 8D. The cases when $(m, n)=(0, 0)$, $(1, 0)$, $(0, 1)$, $(2, 0)$, $(1, 1)$, $(0, 2)$, $(3, 0)(2, 0)$, $(1, 2)$, $(0, 3)$, $(5, 0)$, $(4, 1)$, $(3, 2)$, $(2, 3)$, $(1, 4)$, $(0, 5)$ are covered. A

similar table in the case $\alpha=1$ was given by Sternberg, Shipman and Thurston [same J. **7** (1954), 505–511; MR **16**, 750]. The table was computed on an IBM card Programmed Calculator by methods similar to those used in the $\alpha=1$ case. It is stated that each entry should be correct to within 10^{-8}. Linear interpolation is good to about 4D; a five point Lagrangian interpolation will be good to about 3×10^{-8}.

Some recurrence relations are given, as well as expressions for the functions as power series or in terms of $E(k)$, $K(k)$.

Various occurrences in other fields, either of the Bennett functions themselves, or of functions (e.g. some Weber-Schafheitlin integrals) expressible in terms of these, are noted. *John Todd* (Washington, D.C.).

Binnie, A. M.; and Miller, J. C. P. Tables of two functions required in certain attenuation problems. Quart. J. Mech. Appl. Math. **8** (1955), 468–479.

Tables are provided of the functions

$$R(X,T)=e^{-T}I_0\{(T^2-X^2)^{\frac{1}{2}}\}+2\int_X^T e^{-w}I_0\{(w^2-X^2)^{\frac{1}{2}}\}dw$$

and
$$2\int_X^T R(X,w)dw,$$

which arise in connection with a study of pressure surges in a pipe conveying a viscous liquid [cf. A. M. Binnie, same J. **4** (1951), 330–343]. and in connection with surges in transmission lines (with zero leakage).

The tables were computed at the National Physical Laboratory under the direction of G. F. Miller and give R and Q for $X=0(.2)5$, $T=X(.2)5(1)20$, to 4 or 5S. Second differences (modified where necessary) are provided where linear interpolation would be unsatisfactory. $R(X,T)$ was computed from an expression of the integral occurring in it as a series involving $I_n(T)$ and using tables of this function [Bickley, Comrie, Miller, Sadler, and Thompson, Bessel functions, part 2, Cambridge, 1952; MR **14**, 410]. Q was obtained by a straight-forward quadrature; initial values and checks were taken from series obtained by integrating that used in the computation of R.

There is a worked example.

John Todd (Washington, D.C.).

Beer, A. C.; Chase, M. N.; and Choquard, P. F. Extension of McDougall-Stoner tables of the Fermi-Dirac functions. Helv. Phys. Acta **28** (1955), 529–542.

The Fermi-Dirac functions $F_k(\eta)=\int_0^\infty x^k(1+e^{x-\eta})^{-1}dx$ have been tabulated by J. McDougall and E. C. Stoner [Philos. Trans. Roy. Soc. London. Ser. A. **237** (1938), 67–104], essentially for $k=-1/2, +1/2, +3/2$. In virtue of the relation $(\partial/\partial\eta)F_k(\eta)=kF_{k-1}(\eta)$ it is possible to obtain F_k for higher values of k by numerical integration, using the McDougall-Stoner tables. This was done, by means of the relation

$$\int_0^1 f(x)dx \doteq \tfrac{1}{2}[f(0)+f(1)]-\tfrac{1}{12}[f'(1)-f'(0)] +\tfrac{1}{720}[f'''(1)-f'''(0)],$$

using punched card equipment. The table, in which the McDougall-Stoner table has been incorporated, gives $F_k(\eta)$, to 5S, for $k=-\frac{1}{2}(1)11/2$ and for $k=-4(.1)20$. The initial values were obtained from a power series and checks were obtained at $\eta=0$ from $F_k(0)=(1-2^{-k})k!\zeta(k+1)$ and from the asymptotic expansions at $\eta=10(2)20$. There are small tables giving the coefficients in the asymptotic expansion and the discrepancies between the tabulated values and the check values. These suggest that for $k\geqq3/2$, the tables are correct to 5S and that for $\eta>0$ the errors should be at most a unit in the sixth digit. References to earlier tables are given. *John Todd.*

Sconzo, Pasquale. Tavola per il calcolo della derivata di una funzione data numericamente. Mem. Soc. Astr. Ital. (N.S.) **26** (1955), 393–398.

A table is given of the coefficients $L'_i(t)$ in the differentiated Lagrange formula

$$f'(x_0+th)\doteq h^{-1}\sum_{i=-2}^{2}L'_i(t)f(x_0-ih)$$

for calculating the derivative of a function tabulated at interval h, in terms of five of its values. The table is to 5D, for $t=0(.01)1$ and first differences are given. It was computed by building up $(12)^{-1}L'_i$ from its leading differences and checked by the relation $\sum L'_i(t)=0$. This table is essentially included in one of H. E. Salzer [Nat. Bur. Standards Appl. Math. Ser. no. **2** (1948); MR **10**, 69]. Salzer tabulates $(12)^{-1}L'_i$ exactly for $t=0(.01)2$, as well as similar quantities for the 4-, 6- and 7-point cases.

John Todd (Washington, D.C.).

See also: Freund, p. 571; Jarden and Katz, p. 585; Lah, p. 585; Gupta and Luthra, p. 587; Grad and Solomon, p. 634; Fieller, Lewis and Pearson, p. 638.

Mathematical Machines

★ **Richards, R. K. Arithmetic operations in digital computers.** D. Van Nostrand Company, Inc., Toronto-New York-London, 1955. v+397 pp.

Marczyński, R. Electronic automatic digital computers. Zastos. Mat. **2** (1955), 263–296. (Polish. Russian and English summaries)

Weik, Martin H. A survey of domestic electronic digital computing systems. Ballistic Research Laboratories, Aberdeen Proving Ground, Md., Rep. no. **971** (1955), iv+vii+272 pp. (Government Agencies, their contractors and others cooperating in Government research may obtain reports directly from the Ballistic Research Laboratories. All others may purchase photographic copies from the Office of Technical Services, Department of Commerce, Washington 25, D. C.)

The following quotation describes this report: gives "the engineering characteristics, logical features, operating experiences, cost factors and personnel requirements of eighty-four different domestic digital electronic computing systems are described. An analysis of the computer field, a discussion of trends and a complete glossary of computer engineering and programming terminology are included". In view of the importance of the work, it is unfortunate that "Although the greatest possible care was exercised, the probability that at least one error exists in each computing system described is rather high, ···". It is surprising, however, that for so widely known a machine as the IBM-650 there are obviously erroneous figures, such as 11-decimal digit word length and two orders per word. *R. W. Hamming.*

★ **Computer development (SEAC and DYSEAC) at the National Bureau of Standards, Washington, D.C.** National Bureau of Standards Circular 551. U.S. Government Printing Office, Washington 25, D.C., 1955. iv+146 pp. $2.00.

This volume contains the following papers: Introduction (pp. 1–3) by S. N. Alexander; SEAC (pp. 5–26) by S. Greenwald, S. N. Alexander and Ruth C. Haveter; Dynamic circuitry techniques used in SEAC and DYSEAC (pp. 27–38) by R. D. Elbourn and R. P. Witt; DYSEAC (pp. 39–71) by A. L. Leiner, S. N. Alexander and R. P. Witt; System design of the SEAC and DYSEAC (pp. 73–92) by A. L. Leiner, W. A. Notz, J. L. Smith and A. Weinberger; High-speed memory development at the National Bureau of Standards (pp. 93–108) by R. J. Slutz, A. W. Holt, R. P. Witt and D. C. Friedman; Input-output devices for NBS computers (pp. 109–118) by J. L. Pike and E. F. Ainsworth; Operational experience with SEAC (pp. 119–136) by J. H. Wright, P. D. Shupe, Jr., and J. W. Cooper; SEAC-Review of three years of operation (pp. 137–146) by P. D. Shupe, Jr., and R. A. Kirsch.

Comét, Stig. Notations for partitions. Math. Tables Aids Comput. **9** (1955), 143–146.

Three methods of storing partitions in an electronic computer which are based on the Young diagram are described. From the Hardy-Ramanujan asymptotic formula for $p(n)$ one expects that the number of binary digits required to represent all partitions of n be $O(\sqrt{n})$, and this ideal is actually attained in the third method, the others each requiring $O(n)$ binary digits. The author mentions an application to the machine calculation of the characters of certain symmetric groups.

M. Newman (Washington, D.C.).

Samson, Edward W.; and Mueller, Rolf. Circuit minimization: minimal and irredundant Boolean sums by alternative set method. Communications Laboratory, Electronics Research Directorate, Air Force Cambridge Research Center, Cambridge, Mass. Tech. Rep. **55-109** (1955), v+11 pp.

Samson, Edward W.; and Mueller, Rolf K. Circuit minimization: sum to one process for irredundant sums. Communications Laboratory, Electronics Research Directorate, Air Force Cambridge Research Center, Cambridge, Mass. Tech. Rep. **55-118** (1955), vii+16 pp.

The problem of simplifying switching circuitry is closely allied with the problem of simplifying truth functions, and has given rise to investigations of Boolean functions and Boolean matrices. The authors, beginning with the analysis by Quine [Amer. Math. Monthly **59** (1952), 521–531; MR **14**, 440], have developed an algorithm which tests whether a Boolean function is equal to one. This 'sum-to-one' process is based upon the following theorem: If a_j, b_k, c_l and R are Boolean functions not involving the Boolean variable s explicitly, then

$$s\sum_j a_j+\bar{s}\sum_k b_k+\sum_l c_l \geq R$$

if and only if $\sum_{j,k} a_j b_k+\sum_l c_l \geq R$.

This sum-to-one algorithm is then applied to find all minimal and irredundant Boolean sums equivalent to a given Boolean Function. It seems to the reviewer that the method of application of the basic algorithm to the main problem could be considerably clarified by, for example, attempting to program it for some automatic computer.

S. Gorn (Philadelphia, Pa.).

Henrici, Peter. A subroutine for computations with rational numbers. J. Assoc. Comput. Mach. 3 (1956), 6–9.

Henrici, Peter. Automatic computations with power series. J. Assoc. Comput. Mach. 3 (1956), 10–15.

Reynolds, George E. A new method for extracting square root on desk calculators. Antenna Laboratory, Electronics Research Directorate, Air Force Cambridge Research Center, Cambridge, Mass. Rep. **AFCRC-TR-54-120**, vii+32 pp. (1955).

Choudhury, A. K. The isograph- an electronic root finder. Indian J. Phys. **29** (1955), 468–473.

Hoffmann, Hans. Aufbau und Wirkungsweise neuzeitlicher Integrieranlagen. I. Elektrotech. Z. **77** (1956), 41–52.

Berry, F. R., Jr. Electrical analog solution of certain non-linear problems in vibrations and elastic stability. Proc. Soc. Exper. Stress Anal. **13** (1955), no. 1, 1–12.

Fil'čakov, P. F. On electro-modelling of problems of free streamline flow. Dopovidi Akad. Nauk Ukrain. RSR **1955**, 440–443. (Ukrainian. Russian summary)

Wadel, Louis B. Simulation of digital filters on an electronic analog computer. J. Assoc. Comput. Mach. 3 (1956), 16–21.

See also: Bellman, p. 632.

ASTRONOMY

Zagrebin, D. V. Theory of the regularized geoid. Trudy Inst. Teoret. Astr. **1** (1952), 87–222. (Russian)

The theory is given to determine the geoid with respect to an ellipsoid with three unequal axes. It requires the use of Lamé functions. In applications the geoid is identified with an ellipsoid with three unequal axes and is compared with an ellipsoid of revolution (ellipsoid of Krassovsky: great axis 6378245 m, ellipticity 1/298.3, used now in USSR).

Tables are given for gravity acceleration on the ellipsoid used by the Central Scientific Research Institut for Geodesy, Air Surveying and Mapping, on the ellipsoid of Krassovsky and for the reduction of gravity on the international ellipsoid to those on the first two figures.

W. Jardetzky (New York, N.Y.).

Kolbenheyer, T. Beitrag zur Lösung des Stokesschen Problems für ein dreiachsiges Niveauellipsoid. Mat.-Fyz. Časopis. Slovensk. Akad. Vied **5** (1955), 172–192. (Slovak. Russian and German summaries)

In order to find the gravity acceleration on the ellipsoid the potential is taken in the form of a sum of the potential of the centrifugal force and a linear combination of three

functions

$$I=\int_\lambda^\infty \frac{du}{R^{\frac{1}{2}}}\quad I_1=-\frac{\partial I'}{\partial B}\quad I_2=-\frac{\partial I'}{\partial C}.$$

In these expressions $R=(A+u)(B+u)(C+u)$,

$$I'=\int_\lambda^\infty\left[\frac{x^2}{A+u}+\frac{y^2}{B+u}+\frac{z^2}{C+u}-1\right]\frac{du}{R^{\frac{1}{2}}}$$

and A, B, C are the squares of the ellipsoid axes.
W. Jardetzky (New York, N.Y.).

Schatzman, Evry; et Bel, Nicole. Instabilité d'une masse fluide étendue. C. R. Acad. Sci. Paris **241** (1955), 20–22.

An infinite perfect gas of constant density and pressure is in differential rotation. Conditions for first-order gravitational instability are obtained under a perturbation in a plane perpendicular to the axis of rotation with notice made of the case corresponding to the sun in our galaxy. *R. G. Langebartel* (Urbana, Ill.).

Bel, Nicole. Instabilité d'une masse fluide étendue. C. R. Acad. Sci. Paris **241** (1955), 163–164.

Die Arbeit bildet die Fortsetzung einer Note gleichen Titels von Verf. und E. Schatzmann [siehe das vorstehende Referat] und behandelt die Bedingungen für die Instabilität eines mit der (nicht notwendig konstanten) Winkelgeschwindigkeit $\omega(r)$ rotierenden Mediums bei sinusförmigen Störungen der Dichte und der Relativgeschwindigkeit. Gegenüber den Annahmen der ersten Note ist hier die zur Rotationsachse parallele Komponente der Störungsgeschwindigkeit nicht notwendig von Null verschieden. *K. Maruhn* (Dresden).

Agostinelli, Cataldo. Sulla compatibilità di una forma ellissoidale a tre assi per una massa fluida cosmica rotante, elettricamente conduttrice, immersa in un campo magnetico uniforme. Boll. Un. Mat. Ital. (3) **10** (1955), 17–23.

It is shown that a homogeneous electrically conducting rotating mass of incompressible fluid in a uniform magnetic field can take the form of an ellipsoid with three different axes. *S. Chandrasekhar.*

Hershman [Geršman], B. N.; and Ginsburg [Ginzburg], V. L. Influence of magnetic field on convective instability in the atmospheres of stars and in the ionosphere of the earth. Astr. Ž. 32 (1955), 201–208. (Russian. English summary)

It is pointed out that in certain astrophysical and geophysical connections one must allow the electrical and the thermal conductivities to depend on direction. The theory of the inhibition of convection by a magnetic field [S. Chandrasekhar, Phil. Mag. (7) **43** (1952), 501–532; MR **14**, 813] is revised to take into account these effects. A discussion of the orders of magnitude of the various quantities involved suggest that the inhibition will be ineffective in the applications contemplated.
S. Chandrasekhar (Williams Bay, Wis.).

Elsen, H.; et Ledoux, P. Sur l'application de la méthode de Rayleigh-Ritz à la détermination des fréquences d'oscillations radiales d'étoiles gazeuses à grande concentration massique. Bull. Soc. Roy. Sci. Liège **24** (1955), 239–253.

The equation for the small radial oscillations of a star, with radiation pressure taken into account, is considered. The modes of vibration are determined by the method of Rayleigh-Ritz in which the relative displacement at radius r is expanded as a power-series in even powers of r. Two stellar models are solved in detail and it is shown that the method, which is one of successive approximations, gives quickly converging values for the vibration frequencies. It is concluded that models with insignificant outer convective zones will fit classical Cepheids, but that RR Lyrae stars would require models with large convective zones. The results also indicate that the periods of the fundamental and first harmonic vibrations are in the ratio of 2 to 1. *G. C. McVittie* (Urbana, Ill.).

Pogorzelski, W. Problème du mouvement stationnaire dans une couche gazeuse rayonnante. Ann. Polon. Math. **1** (1955), 367–379.

A plane-stratified stellar atmosphere is considered, the gas being in motion perpendicular to the planes of stratification. The motion of the gas takes place under the gas-pressure and the gravitational force, the equation of continuity is satisfied and the equation of energy contains a term arising from the assumption that polychromatic radiation is traversing the stellar atmosphere in all directions. It is shown that a steady state of motion is possible, the gas moving perpendicularly to the planes of stratification. The solution of this problem depends on the use of a system of non-linear integral equations, and a detailed analysis is provided. *G. C. McVittie.*

Sen, K. K. An estimate of the optical thickness of a spherically symmetric, non-conservative scattering atmosphere. Proc. Nat. Inst. Sci. India. Part. A. 21 (1955), 241–243.

Jefferies, J. T. On the diffusion of radiation from a point or a line source in an infinite medium. Opt. Acta 2 (1955), 109–111.

The following two problems in radiative transfer are solved. (1) A point source emitting radiant energy at a rate $4\pi Q$ is imbedded in an infinite homogeneous medium which scatters radiation isotropically with an albedo $\tilde{\omega}_0$. The corresponding equation of transfer is solved in the so-called Eddington-Milne approximation (which is equivalent to expanding the intensity in Legendre polynomials and retaining only the terms in P_0 and P_2). (2) An infinite line source emitting energy at the rate $4\pi Q$ per unit length is imbedded in a medium with the same properties as in (1) above. The solution for this problem is obtained in the same approximation and involves quadratures over Bessel functions. In both cases the solution for the mean intensity is tabulated for λ $(=1-\tilde{\omega}_0)=1/12$, 1/75 and 1/300. *S. Chandrasekhar* (Williams Bay, Wis.).

RELATIVITY

Infeld, Leopold. Einige Bemerkungen über die Relativitätstheorie. Ann. Physik (6) **16** (1955), 229–240.

This is a discussion of invariance in classical and relativistic mechanics, almost without equations. Special attention is paid to the significance of coordinates; an exposition of the two-body problem in general relativity shows the irrelevance of Fock's harmonic coordinate condition [Acad. Sci. U.S.S.R. J. Phys. **1** (1939), 81–116; MR **1**, 183] at the Newtonian and first post-Newtonian stages of approximation, the essential assumption being

the EIH approximation scheme for the metric tensor [cf. L. Infeld, Acta Phys. Polon. 13 (1954), 187–204; MR **16**, 531].

The paper is complementary to an earlier paper by the same author [Canad. J. Math. 5 (1953), 17–25; MR **14**, 806]. *F. A. E. Pirani* (London).

Tauber, G. E. On equations of motion in general relativity. Canad. J. Phys. 33 (1955), 824–827.

The author generalizes for general relativity Dirac's method [Proc. Roy. Soc. London. Ser. A. 212 (1952), 330–339; MR **14**, 228] for derivation of Lorentz's equations of motion and Maxwell's field equations for a charged particle in an electromagnetic field in the presence of a gravitational field from one variational principle. He goes from the action in the form

$$L=G+\frac{u}{4}F_{\mu\nu}F^{\mu\nu}+\frac{\varkappa\lambda}{2}(v_\mu v^\mu-1),$$

where $G=\frac{1}{2}g^{\mu\nu}[\Gamma_{\mu\lambda}{}^{\sigma}\Gamma_{\nu\sigma}{}^{\lambda}-\Gamma_{\mu\nu}{}^{\sigma}\Gamma_{\mu\nu}{}^{\tau}]$, $\Gamma_{\mu\nu}{}^{\lambda}$ are the Christoffel symbols, and

$$F_{\mu\nu}=\frac{\partial A_\mu}{\partial x_\nu}-\frac{\partial A_\nu}{\partial x^\mu}$$

are derived from the potentials $A_\mu=-kv_\mu+\xi\partial\eta/\partial x^\mu$. Taking as independent variables v_μ, λ, ξ, η, $g^{\mu\nu}$ (the notations are the same as in Dirac's paper) and varying the action integral

$$I=\int L\sqrt{-g}\,d\tau_4$$

over the whole of space-time, under the condition that it should be stationary, he obtains four sets of equations. From these he then obtains Maxwell's field equations and the equations of motion. *T. P. Anđelić* (Belgrade).

Møller, C. Old problems in the general theory of relativity viewed from a new angle. Danske Vid. Selsk. Mat.-Fys. Medd. 30 (1955), no. 10, 29 pp.

It is well-known that in general relativity theory the rate of an ideal standard clock moving with the velocity v through a gravitational field with the potential χ is determined by the formula $d\tau=dt(1+2\chi c^{-2}-v^2c^{-2})^{\frac{1}{2}}$, where τ is the proper time of the standard clock and t is the coordinate time. The purpose of this paper is to justify the above formula on the basis of the dynamical laws governing the functioning of a simple model of a clock. A real clock can, as a careful analysis shows, only approximately be considered ideal and its degree of accuracy depends on the properties of the gravitational field in which the clock is placed. The "atomic clock" can be regarded with great accuracy as ideal. *L. Infeld* (Warsaw).

★**Finzi, Bruno. Relatività generale e teorie unitarie.** Cinquant'anni di Relatività, 1905–1955, pp. 135–306. Editrice Universitaria, Firenze, 1955.

A wide-ranging survey article with a mathematical bias, readable by anyone with a slight previous knowledge of tensor calculus. Newtonian physics and tensor calculus in 3-space: 27 pages. Special relativity: 22 pages. Riemannian tensor calculus and General relativity: 63 pages. Cosmology: 13 pages. Unified theories: 26 pages. General remarks: 12 pages. The only unusual feature is an 8-page exposition of the author's work showing that the characteristic surfaces of the gravitational field are null cones [Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) 6 (1949), 18–25; MR **10**, 747]. *F. A. E. Pirani.*

Castoldi, Luigi. Relatività Riemanniana unitaria. Rend. Sem. Fac. Sci. Univ. Cagliari 25 (1955), 44–62.

The author develops further the ideas introduced by him in an earlier paper [Atti Accad. Ligure 9 (1952), 5–14; MR **15**, 169]. In this paper he starts with the equations

$$G_i{}^j\equiv R_i{}^j-\frac{1}{2}a_i{}^jR=\chi(T_i{}^j-E_i{}^j),$$

where $T_i{}^j=\sigma u_iu^j$ and

$$E_i{}^j=\frac{1}{4}(F_{ih}F^{hj}+\frac{1}{4}a_i{}^jF_{hk}F^{hk})/\pi,$$

χ being a constant and σ a scalar (taken to be unity when u_i, instead of being a unit vector, is null). Given a Riemannian space R_4 of fundamental tensor a_{ij}, these may be regarded as equations for the unknowns u_i, F_{ij}, the latter being by hypothesis a 6-vector. Conversely, if u_i, F_{ij} are regarded as given in an initially amorphous manifold V_4, then the equations may be used to construct an Einstein tensor $G_i{}^j$ and hence to obtain a metric tensor rendering the manifold Riemannian. The electromagnetic field and the geometry are thus inextricably bound up with one another, a situation characteristic of a unified theory. Attention becomes focussed upon the divergence-equation

$$\nabla_j(T_i{}^j-E_i{}^j)=0,$$

which is a necessary consequence of the identity $\nabla_jG_i{}^j=0$. By laying down certain postulates chosen for their physical naturalness or because they conform to the principle of simplicity, the author is able to argue that the laws of the electromagnetic and gravitational fields are contained in the divergence-equation alone. The consequences of his postulates are worked out in some detail. *H. S. Ruse* (Leeds).

Renaudie, Josette. Théorie unitaire à six dimensions. Equations du champ. C. R. Acad. Sci. Paris 240 (1955), 399–401.

Renaudie, Josette. Théorie unitaire à six dimensions. Interprétation pour le champ mésonique-électromagnétique. C. R. Acad. Sci. Paris 240 (1955), 2380–2382.

Pour obtenir une théorie unitaire à six dimensions, on suppose en général qu'un espace riemannien V_6 à six dimensions admet un groupe de transformations à deux paramètres. J. Podolanski [Proc. Roy. Soc. London. Ser. A. 201 (1950), 234–260; MR **11**, 746] suppose que V_6 admet deux translations commutative orthogonales. K. Yano et M. Ohgane [Rend. Mat. e Appl. (5) 13 (1954), 99–132; MR **16**, 184] suppose que V_6 admet un groupe général d'isométries à deux paramètres. Ici l'auteur suppose que V_6 admet un groupe d'isométries abélien à deux paramètres.

On suppose que le groupe est engendré par deux vecteurs de Killing $\boldsymbol{\xi}$ et $\boldsymbol{\eta}$. Soient L_0 les trajectoires de l'isométrie ξ. On considère l'espace-quotient V_5 de V_6 par L_0. V_5 admet un groupe d'isométries à un paramètre. Donc on peut considérer un espace-quotient V_4 de V_5 par les trajectoires du vecteur de Killing dans V_5. V_4 jouera le rôle de l'espace-temps. L'auteur calcule le tenseur de Ricci de V_6 et établit les équations du champs. On trouve ici trois tenseurs: le premier est interprété comme champ gravitationnel, le second comme tenseur électromagnétique et le dernier comme tenseur mésonique.

A la fin de la seconde Note, l'auteur étudie le problème de Cauchy relatif à ces équations de champs.

Le travail doit avoir une relation étroite avec celui de B. Hoffmann [Phys. Rev. (2) 73 (1948), 30–35; MR **9**, 387]. *K. Yano* (Tokyo).

Brinis, Elisa. Qualche illustrazione geometrica dello spazio unitario di Einstein. Ist. Lombardo Sci. Lett. Rend. Cl. Sci. Mat. Nat. (3) **19(88)** (1955), 531–538.

This is a review of some simple geometric properties of the two parallel displacements occurring in the non-symmetric metric connexion in Einstein's unitary field theory. *J. A. Schouten* (Epe).

Takasu, Tsurusaburo. Non-conjectural theory of relativity as a non-holonomic Laguerre geometry realized in the three-dimensional torsioned Cartesian space fibered with actions. Proc. Japan Acad. **31** (1955), 606–609.

This is another attempt to relate the theory of relativity with some geometry of a Laguerre type. *J. A. Schouten* (Epe).

Littlewood, D. E. The behaviour of the universe. Proc. Cambridge Philos. Soc. **52** (1956), 88–96.

The metrics and other properties of uniform models of the universe, in which the pressure is zero, are obtained by the author's method of conformal transformations. The results are not new but the author claims that his method is preferable to the usual one because the tensor calculus is not employed. *G. C. McVittie* (Urbana, Ill.).

Haywood, J. H. The equations of motion of rotating bodies in general relativity. Proc. Phys. Soc. Sect. A. **69** (1956), 2–15.

The Fock-Papapetrou approximation scheme [V. A. Fock, Acad. Sci. U.S.S.R. J. Phys. **1** (1939), 81–116; MR **1**, 183; A. Papapetrou, Proc. Phys. Soc. Sect. A. **64** (1951), 57–75, 302–310; MR **12**, 546; **13**, 695] is applied to the problem of two slowly-moving rotating spheres in a weak slowly-varying gravitational field, with spheres which are (a) spherically symmetric, (b) fluid, and so spheroidal. The calculations, which are very involved, are carried to order $(V/C)^2$. *F. A. E. Pirani* (London).

Clauser, Emilio. Propagazione della luce nei cristalli in moto. Ist. Lombardo Sci. Lett. Rend. Cl. Sci. Mat. Nat. (3) **19(88)** (1955), 287–320.

Let there be a homogeneous crystal, magnetically isotropic and electrically anisotropic with dielectric tensor η_{ik}, at rest in an inertial frame of reference. If the space axes are principal axes of η_{ik}, the partial differential equation of the characteristic surfaces (or wave surfaces) can be simply expressed, as shown by T. Levi-Civita [Caratteristiche dei sistemi differenziali e propagazione ondosa, Zanichelli, Bologna, 1931, par. 9]. To study the waves as viewed by an observer moving relative to the crystal, the author applies a Lorentz transformation to the above partial differential equation. Since the original space axes are tied to the crystal, the Lorentz transformation cannot be simplified by choice of axes, and in consequence the algebra is heavy. The theory is discussed in great detail, and the author finds two extreme cases: in the first, the crystal is doubly refracting in all directions (as for the crystal at rest) and, in the second, it is doubly refracting in some directions and quadruply refracting in others. *J. L. Synge* (Dublin).

Romain, Jacques. Théorie du choc élastique de particules dans l'espace-temps de Minkowski. Acad. Roy. Belg. Bull. Cl. Sci. (5) **41** (1955), 1225–1241.

The author views the collisions of particles geometrically in Minkowskian space-time. He discusses a number of problems of collision, annihilation and creation of material particles and photons, using only the laws of conservation of momentum and energy, which laws tell us that the total 4-momentum of the system is the same before and after. *J. L. Synge* (Dublin).

MECHANICS

Tainoŭ, A. I. Kinematics of Assur's groups of second class and second type. Vesci Akad. Navuk Belarusk. SSR. **1954**, no. 1, 183–186. (Belorussian)

The plane mechanism considered consists of a closed four-link hinged chain $M_1M_2M_3M_4$ and four bars ("reins") A_iB_i hinged to M_i at B_i. The problem is to find the velocity distribution in the links M_i when the velocities of the points A_i are given. The method is based on the following lemma: If the points P and Q of two hinged links are in positions collinear with the hinge, their velocities have the same projections on the (oriented) line PQ. This lemma is credited in the Russian literature to L. V. Assur (1916). [It can be found, for instance, in Rosenauer and Willis, Kinematics of mechanisms, Associated general publications, Sydney (Australia), 1953, p. 299. This book considers many problems similar to that of the paper, and uses similar methods.] It follows that if a point P_i of M_i is simultaneously in a position collinear with a point of M_k and the hinge joining M_i and M_k for two values of k, the velocity of P_i is determined by the two velocities of the points P_k. A successive application of this procedure leads to the solution of the problem. *A. W. Wundheiler.*

Müller, Hans Robert. Zur Kinematik des Rollgleitens. II. Sphärisches Rollgleiten. Arch. Math. **6** (1955), 471–480.

Man spricht von Rollgleiten wenn zwei ebene Kurven C und C' sich ständig berühren und der Berührungspunkt C und C' mit Geschwindigkeiten durchläuft, die im konstanten Verhältniss λ stehen [Grüss, Z. Angew. Math. Mech. **31** (1951), 97–103; MR **13**, 80; H. R. Müller, Arch. Math. **4** (1953), 239–246; MR **15**, 171]. Die Betrachtungen werden jetzt auf die sphärische Kinematik erweitert. Die bestimmende Differentialgleichung kann jetzt nicht elementar integriert werden; sie ist auf eine Riccatische Gleichung zurückführbar. Anwendung auf das sphärische Verzahnungsproblem. Mittels des Studyschen Übertragungsprinzipes wird noch eine Erweiterung des Rollgleitens in der Kinematik des Linienraumes formuliert. *O. Bottema* (Delft).

Kestelman, H. Finite rotations of a rigid body. Math. Gaz. **39** (1955), 278–279.

A rigid body is freely pivoted at a point O through which pass three fixed axes OA, OB, OC; if the body is given one finite rotation about each of these, in the order stated, what conditions must be satisfied by the angles between the axes if the rotations are to be capable of moving the body into every position? The necessary and sufficient condition is proved to be: OB is perpendicular to OA and to OC. *O. Bottema* (Delft).

Boerdijk, A. H. The mechanical equilibrium of a body rolling on a plane. Simon Stevin **30** (1955), 193–231. (Dutch. English summary)

A rigid body B is placed upon a fixed surface O in a

position of equilibrium with respect to gravity and its stability discussed for small rolling of B on O. Sufficient conditions for stability are given for the case of a plane rigid body and expressed in the radii of curvature of B and O in the contact point. For instance: if B is a rectangle Z its center of gravity, A the point of contact, O a circle of radius r, then the condition of stability reads $ZA<r$. Special attention is given to the cases of indifferent equilibrium. The sphere on a horizontal plane is well-known, but there are other cases e.g. a logarithmic spiral on an inclined plane. Application to cams and gears. Extension to space kinematics. *O. Bottema* (Delft).

Mihăilă, N. Sur les déplacements infinitésimaux des systèmes matériels indéformables. Gaz. Mat. Fiz. Ser. A. 7 (1955), 428–433. (Romanian. Russian and French summaries)

L'auteur expose d'une manière synthétique un procédé vectoriel permettant l'étude des déplacements infinitésimaux. La rigidité du système est caractérisée par le fait que tout vector de module constant subit une variation donnée par la relation $\overline{dr}=\overline{d\theta}\times\bar{r}$. *O. Bottema* (Delft).

Silaş, Gheorghe. Sur certains théorèmes de mécanique. Gaz. Mat. Fiz. Ser. A. 7 (1955), 434–438. (Romanian. Russian and French summaries)

L'accélération d'un point M, en mouvement sur une conique sous l'action d'une force centrale est $k\varrho/u^3$, où k est une constante, ϱ la distance de M au centre A et u la distance de M à la polaire de A par rapport à la conique. *O. Bottema* (Delft).

Efimov, M. I. On equations of motion in holonomic and nonholonomic parameters. Prikl. Mat. Meh. 19 (1955), 762–764. (Russian)

According to the author, M. F. Shul'gin [Prikl. Mat. Meh. 18 (1954), 749–752; MR 16, 873], "in considering the problem of exposing certain errors" in a previous paper of the author [ibid. 17 (1953), 748–750; MR 15, 659] "accomplished this with insufficient accuracy".
A. W. Wundheiler (Chicago, Ill.).

de Varennes e Mendonça, P. Le principe gaussien de la moindre contrainte n'est pas exact. Portugal. Math. 14 (1955), 73–77.

It is shown that if the fictitious constrained motion agrees with the actual motion in both velocity and acceleration at the given instant, then Gauss's principle of least constraint may fail to hold. An analysis of the terms making up the constraint exposes conditions which are sufficient for the validity of the principle. *H. D. Block.*

Bressan, Aldo. Sull'impossibilità dinamica di un certo tipo di precessioni. Rend. Sem. Mat. Univ. Padova 24 (1955), 396–399.

The type of precession referred to in the title is that of a heavy asymmetric rigid body fixed without friction at a point on one of its principal inertial axes (but not coinciding with the center of gravity). The author shows the impossibility of precession about a horizontal axis with the axis of nutation equal to $\pi/2$. *D. C. Lewis.*

Hamilton, H. J. Uniform circular motion is singular. Amer. Math. Monthly 63 (1956), 109–111.

Rivlin, R. S. Plane strain of a net formed by inextensible cords. J. Rational Mech. Anal. 4 (1955), 951–974.

Mechanics of a plane net formed by two families of parallel inextensible cords, which are straight in the undeformed state of the net. The author makes some simplifying assumptions thus idealising a certain type of textile frabric and then considers the pure homogeneous deformation of such a sheet treated as a plane continuum, putting the condition that line-elements of two fixed directions do not change their lengths. He proves that then the principal directions of the deformation bisect the angles between the cords and that the displacements of the various points of the sheet satisfy a simple partial differential equation of hyperbolic type. This is solved for some types of boundary conditions. Stress-deformations are discussed and it is shown how these may be used in order to determine the deformation which results from the application of specified edge tractions to a sheet of arbitrary contour. *O. Bottema* (Delft).

Manacorda, Tristano. Una osservazione sulla dinamica dei fili. Boll. Un. Mat. Ital. (3) 9 (1954), 385–390.

The following review replaces an earlier one [MR 16, 874] of this paper.

It is shown that if a perfectly flexible and elastic cord AB is constrained to plane vertical motion, and if the load is vertical, the displacement $z(s,t)$ of a particle $P(s)$ of the cord (s is the natural length of the cord AP) can be expressed by means of quadratures in terms of three functions α, ϕ, θ, of one variable. $\alpha(s)$ is the initial horizontal abscissa of $P(s)$, $\phi(t)=z(0,t)$, and $\theta(t)$ the horizontal component of the tension of the cord. Under the assumptions made, the equation of horizontal motion states that $\theta(t)$ is independent of s, and the other differential equation is not necessary to derive the expression for $z(s,t)$.

The relevance of the assumption that θ is known is not discussed. The notation of the paper is very redundant. For instance, $u(s)$ is written for zero, $x(s)$ for $\alpha(s)$, $-\phi$ is denoted by h, etc. The phrase "funzioni assegnate" is used for sets of dependent functions.
A. W. Wundheiler (Chicago, Ill.).

Auch, Karl; und Braunbek, Werner. Zur Darstellung der Bewegungsformen eines ungedämpften, linearen Schwing-Systems mit zwei Freiheitsgraden. Ann. Physik (6) 15 (1955), 255–267.

Two masses are coupled to each other and to two fixed points lying in their extended straight line by means of three springs. Friction is ignored, and motion along this line is considered. The usual linear differential equations are solved in a standard way getting the usual normal mode solution. This is then given a straightforward graphical representation. *E. Pinney* (Berkeley, Calif.).

Wiebelitz, Rudolf. Zur Theorie der erzwungenen Schwingungen des symmetrischen Kreisels. Z. Angew. Math. Phys. 6 (1955), 362–377.

The author studies the motion of a symmetrical top which is subjected to a constant gravitational field acting in the negative z-direction, and to a force having components $k_2 \cos \omega t$, $k_1 \sin \omega t$, 0 in the x, y, z-directions, respectively. Assuming that the angle between the axis of the top and the z-axis remains small, he linearizes the equations of motion, solves the linearized equations by elementary methods, and gives a full physical discussion of the solution. In one particular case, in which $k_1=k_2$, the nonlinear differential equations of motion can be solved exactly; and this case is discussed in detail. A method of successive approximations is described, and is shown to

lead to a solution in cases in which the angular velocity of the top about its axis is sufficiently large.
L. A. MacColl (New York, N.Y.).

Rachkovitch, Daniel. Quelques propriétés de l'équation caractéristique d'un système mécanique oscillant soumis à liaisons statiques. C. R. Acad. Sci. Paris **242** (1956), 448–449.

Fluid Mechanics, Acoustics

★ **Truesdell, C. The kinematics of vorticity.** Indiana Univ. Publ. Sci. Ser. no. 19. Indiana University Press, Bloomington, 1954. xvii+232 pp. $6.00.

This book is concerned with the purely kinematical properties of the vorticity vector in hydrodynamics (except for some parenthetical remarks on the dynamics of the problem), and the author concentrates on the general aspects of the subject rather than on particular applications.

Chapters I and II are introductory and deal with the general properties of vector fields. The vorticity vector is introduced explicitly in Chapter III and various interpretations of its significance are given, including the familiar one due to Stokes. Regarding the representation of the velocity (or any other vector) in terms of its scalar and vector potentials, the author remarks that this representation is "more important for its implications than for any direct use···". However, remembering that this formula leads to the "law of Biot-Savart" and all its applications the reviewer cannot share the author's view in this respect. Chapter IV deals with circulation preserving motions, including the theorems of Kelvin and Helmholtz. It also includes analyses of the "flexion vector" (the curl of the vorticity) of Beltrami motions (in which the vortex-lines and the stream-lines coincide) and of complex-lamellar motions (stream-lines coinciding with the stream-lines of a field of potential flow). Chapter V is concerned with the comparison of the (local or average) intensities of vorticity and deformation, and two non-dimensional measures of the intensity of the vorticity are defined and analysed.

There follow chapters on vorticity averages and on the properties of the moments of the vorticity; on Bernouilli theorems (with the dynamical variables eliminated to a large extent); and on the convection and diffusion of vorticity. The detailed discussion of the velocity-potential theorem or theorems is of great interest.

Nothing dies in Mathematics and the day may come when even some of the seemingly "useless" parts of the book will find their application. In any case the treatise is of great value as a contemporary "handbook" on the subject. Also, the author's great erudition in the history of Hydrodynamics should not mask the fact that a considerable number of his own contributions is included in the work under review. *A. Robinson* (Toronto, Ont.).

Carstoiu, Ion. Vorticity and deformation in fluid mechanics. A contribution to their kinematical properties. J. Rational Mech. Anal. **3** (1954), 691–712.

This contribution belongs to the same field as the book reviewed above. It contains (i) a counterpart, for the deformation tensor, to Cauchy's well-known formula for the variation of the vorticity from the Lagrangian point of view; (ii) an analysis of the areas swept out by the acceleration vector varying on a curve or surface, where the vector is attached either to the curve (surface) or to the origin; and (iii) various formulae for the motion of material lines and surfaces. As a particular application, the following theorem is obtained. If at each point of a certain vortex line the deformation quadric is degenerate, then the velocity is constant along that line.

Rigid motions are considered by way of illustration. The author states that much of the material in the present paper was contenued in his Thesis [Paris, 1948].
A. Robinson (Toronto, Ont.).

Ericksen, J. L. Note concerning the number of directions which, in a given motion, suffer no instantaneous rotation. J. Washington Acad. Sci. **45** (1955), 65–66.

Dans le mouvement d'un milieu continu le nombre des directions qui ne subissent aucune rotation instantanée est lié au signe d'un discriminant qui a été explicité par Truesdell [dans le livre analysé ci-dessus]. L'Auteur donne ici une interprétation cinématique de ces conditions de signe, et il montre que pour que le nombre de ces directions soit égal à trois il est nécéssaire que $\mathcal{W}_k<1$, $\mathcal{W}_k$ étant l'invariant introduit par Truesdell [J. Rational Mech. Anal. **2** (1953), 173–217; MR **14**, 1026] pour mesurer l'importance relative du tourbillon dans le mouvement.
R. Gerber (Toulon).

Bilimovitch, Anton. On the restitution of homogeneity in equations of velocital character. Glas Srpske Akad. Nauka **206**. Od. Prirod.-Mat. Nauka (N.S.) **5** (1953), 43–48. (Serbo-Croatian. English summary)

The Serbo-Croatian version of Acad. Serbe Sci. Publ. Inst. Math. **5** (1953), 29–34 [MR **15**, 476]. *T. P. Anđelić.*

Verschaffelt, J. E. Sur l'écoulement relatif d'un fluide. Acad. Roy. Belg. Bull. Cl. Sci. (5) **41** (1955), 1106–1112.

Ludford, G. S. S.; Martinek, J.; and Yeh, G. C. K. The sphere theorem in potential theory. Proc. Cambridge Philos. Soc. **51** (1955), 389–393.

Let $\varphi_0(r)\equiv\varphi_0(r,\theta,\psi)$ be the potential of a distribution of total mass zero lying inside the sphere $r=a$. Then

$$\varphi_1(r,\theta,\psi)=\frac{a}{r}\varphi_0\left(\frac{a^2}{r}\right)+\frac{2}{ar}\int_a^\infty\lambda\varphi_0\left(\frac{\lambda^2}{r}\right)d\lambda$$

is harmonic inside and on the sphere and satisfies $\partial(\varphi_0+\varphi_1)/\partial n=0$ on $r=a$, where n denotes the outward normal of the region considered.

The corresponding theorem in which the distribution is exterior to the sphere was given by P. Weiss [same Proc. **40** (1944), 259–261; MR **6**, 191] and is obtainable from the above by writing 0 for ∞ as the upper limit; indeed both theorems depend on the simple fact that inversion interchanges interior and exterior. For this reason also the circle theorem [Milne-Thomson, ibid. **36** (1940), 246–247; MR **1**, 284] applies both ways without modification, contrary to what the authors appear to indicate. The authors give also a generalization of their main theorem above adapted to the boundary condition $\partial(\varphi_0+\varphi_1)/\partial n+h(\varphi_0+\varphi_1)=0$, $h>0$.
L. M. Milne-Thomson (Sevenoaks).

Fraenkel, L. E. On the flow of rotating fluid past bodies in a pipe. Proc. Roy. Soc. London. Ser. A. **233** (1956), 506–526.

This paper deals with the axisymmetric flow of a rotating, inviscid, incompressible fluid through contractions and expansions of a pipe, or past bodies of revolution

on the pipe axis. The flow far upstream is assumed to have constant axial velocity U, and constant angular velocity Ω about the axis. Under this condition the equation governing the stream function becomes linear. Two problems are considered in particular; In problem 1 the flow field is constructed from the stream function of a ring source on the pipe wall; this solution is then used to calculate the effect of a contraction of the pipe. Problem 2 concerns the flow due to a point source or a point doublet on the axis; this solution is further applied to calculate the flow past oval bodies of revolution.

In both problems the ratio $ka\equiv 2\Omega a/U$ (where a is the pipe radius) is an important parameter, of which the zeros j_n of the Bessel function J_1 are critical values. For $ka<j_1$ ($=3.83$), the solutions are unique; while for $ka>j_1$ the solutions involve wave motions, and are not unique. In this latter case uniqueness is obtained by assuming that there are no disturbances far upstream so that the wave motions are permitted only on the downstream side. As to the validity of this assumption, it is argued that the present solutions apply only for a limited range, as yet not quite known, of kl (where l is some characteristic body length). With the above assumption some streamline pictures and velocity distributions are presented, and the physical behavior of a rotating fluid is discussed.
T. Y. Wu (Pasadena, Calif.).

Gheorghiţă, St. I. Le mouvement irrotationnel d'un fluide idéal incompressible en présence d'une sphère poreuse. Acad. R. P. Romîne. Bul. Şti. Secţ. Şti. Mat. Fiz. **7** (1955), 757–762. (Romanian. Russian and French summaries)

The author studies the problem of the irrotational motion of a fluid in the presence of a porous sphere. The velocity-components in the fluid, and in the porous sphere are obtained, as well as the magnitude of the force acting on the sphere. *K. Bhagwandin* (Oslo).

Gheorghiţă, St. I. Généralisation de la formule de Stokes. Acad. R. P. Romîne. Bul. Şt. Secţ. Şti. Mat. Fiz. **7** (1955), 751–756. (Romanian. Russian and French summaries)

The author obtains exact expressions for the velocity and pressure components for the case of the slow motion of a porous sphere in a viscous incompressible fluid. Stokes' formula for the expression of the force-component is generalized. *K. Bhagwandin* (Oslo).

Loĭcyanskiĭ, L. G. The hydrodynamical theory of a ball bearing. Prikl. Mat. Meh. **19** (1955), 531–540. (Russian)

The author obtains approximate expressions for the pressure and moment acting on a rigid sphere situated within an exterior sphere which is filled with viscous fluid. The excentric motion of the inner sphere is considered to be quasi-stationary. The author's solution is more general than that given by Wannier [Quart. Appl. Math. **8** (1950), 1–32; MR **12**, 217]. The author proceeds from the general equations of motion in which some approximations are introduced relative to the particular problem at hand. The partial differential equation for the pressure is then decomposed into three ordinary differential equations. Two of the equations are solved by means of Galërkin's method, while the solution of the third one is elementary. By means of well-known methods, the author obtains finally expressions for the components of the reactive forces, and for the moment acting on the inner sphere. Neither numerical results nor experimental evidences are given, yet the author's analysis seems reasonable.
K. Bhagwandin (Oslo).

Vasilache, Sergiu. Sur certains problèmes de la théorie des infiltrations. Acad. R. P. Romîne. Bul. Şti. Secţ. Şti. Mat. Fiz. **7** (1955), 365–385. (Romanian. Russian and French summaries)

In this paper the author extends the results obtained in previous investigations in the theory of infiltrations [Com. Acad. R. P. Române **1** (1951), 331–336, 551–555; MR **17**, 425]. He studies the problem of the plane movement of incompressible fluids in porous media bounded by polygons. Here again the problem is reduced to the solution of Laplace's equation ($\Delta\varphi(x,y)=0$) for the velocity potential (Darcy-type approximation). The values of $\varphi(x,y)$ are given on certain segments of the contour of the polygon, while for the rest of the boundary the normal derivative $d\varphi/dn$ is given. The author's solution of the problem, by means of conformal transformation, leads to the mixed problem of Volterra [Ann. Mat. Pura Appl. (2) **11** (1882), 1–55] of complex-function theory. The problem consists of determining a holomorphic function $f(z)=\varphi(x,y)+i\psi(x,y)$ uniquely, when $\varphi(x,y)$ is given on some part of the contour, while $\psi(x,y)$ is given on the rest of the contour [cf. A. Signorini, ibid. (3) **25** (1916), 253–273]. The author makes two applications of his theory, viz. a) the infiltration in a region between two lines, b) the movement of an incompressible fluid in a homogeneous and isotropic medium bounded by a step-like basin. After a lengthy analysis, the author obtains solutions of the problems in terms of elliptic functions. [A similar problem has been solved by N. N. Pavlovskiĭ: Teoriya dviženiya gruntovyh vod pod gidrotehničeskimi sooruženiyami i eë osnovnye priloženiya, Petrograd, 1922.] *K. Bhagwandin* (Oslo).

Grodzovskiĭ, G. L. Flow of a viscous gas between two moving parallel plane walls and between two rotating cylinders. Prikl. Mat. Meh. **19** (1955), 99–102. (Russian)

The first part of the paper is devoted to the analysis of the motion of viscous compressible gases, enclosed between two plane parallel walls in motion. The longitudinal pressure gradient dp/dx is absent, while $y=\text{const}$ defines the surface of the fluid. Under these assumptions, the author obtains two ordinary second-order differential equations for the determination of the temperature profile and the velocity distribution. The results are plotted in two figures. The second part of the paper deals with the motion of a fluid between two rotating concentric circular cylinders. The expressions for the temperature profile and the velocity distribution are given in terms of elementary functions, and exhibited by two curves. Apparently, the author does not seem to be familiar with G. I. Taylor's [Philos. Trans. Roy. Soc. London. Ser. A. **223** (1923), 289–343] solutions of the latter type of problems. *K. Bhagawandin* (Oslo).

Pavlin, A. K. On a case of integration of the equations of motion of a viscous fluid with variable coefficient of viscosity. Prikl. Mat. Meh. **19** (1955), 635–638. (Russian)

The author considers the problem of heat conduction in a viscous fluid, contained between two parallel unbounded planes. One of the bounding planes is made to move with constant velocity. The problem is reduced to the solution

of an ordinary third-order differential equation for the velocity component, wherefrom the author derives an expression for the temperature distribution. The results are displayed in graphs and in a table. No references are made to other works in this field. *K. Bhagwandin.*

Binnie, A. M. The effect of viscosity upon the critical flow of a liquid through a constriction. Quart. J. Mech. Appl. Math. 8 (1955), 394–414.

This paper deals with the viscous effects upon the "critical flows" of a liquid with a free surface flowing through two particular boundary constrictions: (1) the laminar flow of a viscous liquid under gravity over a broad-crested weir, and (2) the swirl flow of a viscous liquid under pressure through a convergent-divergent nozzle, with the free surface forming an inner core of the flow. The constrictions are assumed to be gradual so that the transverse components of velocity and acceleration normal to the constrictions may be neglected. The pressure gradients in the stream direction are assumed to be favorable so that the flow does not separate from the constrictions. The crosswise dimension of the flows is assumed to decrease monotonically, and hence the stream velocity to increase monotonically (from the continuity condition) along the direction of streaming. The viscous effects are represented in such a way that a laminar boundary layer extends throughout the whole flow field, the velocity distributions over the cross-sections being taken to be of polynomial forms. Thus, to the reviewer's mind, the present analysis applies only for highly viscous liquids, or, rather for the flows having small values of the Reynolds number of the problem. While this formulation holds in many practical cases [see, e.g., G. I. Taylor, same J. 3 (1950), 129–139; MR **11**, 697], some other cases where the boundary layer extends to only a part of the discharging flow were not treated in this paper. The two flows mentioned above are "critical" in the sense that arbitrary inlet conditions may not be specified, but must be determined from the conditions that prevail at the critical sections. In the case of inviscid flows the critical section is at the crest of the weir, or at the throat of the nozzle. When viscosity is considered, the critical sections are displaced downstream from the crest or the throat. A numerical example is given for each type of motion; their solutions are obtained by a step-by-step integration starting at a suitably chosen location of the critical section together with some appropriate critical conditions.

It is also proved that a long wave of small amplitude can maintain itself stationary on the moving liquid at the critical sections. *T. Y. Wu* (Pasadena, Calif.).

v. Krzywoblocki, M. Z. On the variation of kinetic energy and internal energy of a viscous compressible fluid flow in a two-dimensional domain. Bull. Soc. Math. Grèce 28 (1954), 37–50.

J. Leray [J. Math. Pures Appl. (9) 13 (1934), 331–418] proved that the kinetic energy of an incompressible viscous fluid, contained in a two-dimensional finite domain with fixed boundaries, decreases exponentially with respect to the time t; the reviewer [C.R. Acad. Sci. Paris 223 (1946), 1096–1098; later gave a more refined expression of the exponential bound. The author considers the more general case of a compressible viscous fluid and gives exponential bounds not only for the kinetic energy but also for the internal energy and the density.

J. Kampé de Fériet (Lille).

Shigemitsu, Yutaka. Statistical theory of turbulence. III. Extension of similarity theory of turbulence. J. Phys. Soc. Japan 10 (1955), 1077–1087.

L'auteur applique les résultats qu'il a obtenus précédemment [même J. 10 (1955), 472–482, 890–902; MR 17, 312] à divers problèmes de turbulence. A) Turbulence libre derrière un obstacle. La distribution du facteur d'intermittence de A. A. Townsend suivant la direction Oy s'obtient par intégration d'une fonction de Gauss. L'expérience est en bon accord avec les prévisions théoriques. B) Comparaison de la turbulence longitudinale et de la turbulence transversale. Si $\overline{u^2}$, $\overline{v^2}$, $\overline{w^2}$, $\overline{uv}$ représentent des valeurs moyennes, $u/(u^2+v^2)^{\frac{1}{2}}$ et $\overline{uv}/uv$ ont des valeurs sensiblement constantes dans le fluide, quelle que soit sa nature. Il semble qu'il en soit de même pour w/u. C) Distorsion de la turbulence isotrope. Dans la turbulence sans tensions obliques, $\overline{uv}=0$, mais u et v ne sont pas nécessairement égaux. D) Un exemple simple de turbulence avec tensions est enfin examiné, dans lequel on étudie la distribution de l'intensité de la turbulence et de la vitesse moyenne. *J. Bass* (Paris).

Chamberlain, Joseph W.; and Roberts, Paul H. Turbulence spectrum in Chandrasekhar's theory. Phys. Rev. (2) 99 (1955), 1674–1677.

Dans une récente théorie [Proc. Roy. Soc. London. Ser. A. 229 (1955), 1–19; MR 16, 968] Chandrasekhar a étudié la turbulence homogène, isotrope, stationaire, et a établi, à partir de diverses hypothèses, une équation pour la fonction de corrélation longitudinale $f(r, t)$ où r et t sont des intervalles réduits d'espace et de temps. Les auteurs se proposent de traduire cette théorie en termes spectraux. Ils définissent une fonction spectrale

$$F(k, t)=c_1k^2\{\mathbf{V}_k(t_0)\cdot\mathbf{V}_k(t_0+t)\}_{\mathrm{av}}$$

où $\mathbf{V}_k(t_0)$ est la vitesse au temps t_0 pour un tourbillon de nombre d'ondes k, et où $\{\ \}_{\mathrm{av}}$ est une moyenne spatiale. Ils établissent la relation entre $F(k, t)$ et $f(r, t)$ et en déduisent l'expression de $F(k, t)$ dans le cas des grands nombres de Reynolds. Ils indiquent ensuite comment se transforme l'équation fondamentale de Chandrasekhar dans les deux cas suivants: a) lorsque r et t sont petits (on trouve alors que F vérifie l'équation linéaire:

$$\frac{\partial^2 F}{\partial t^2}=(\nu^2k^2-1)k^2F,$$

dont la solution s'obtient et se discute aisément); b) dans le cas général non linéarisé. *J. Bass* (Paris).

Kraichnan, Robert H. Pressure field within homogeneous anisotropic turbulence. J. Acoust. Soc. Amer. 28 (1956), 64–72.

In this paper general results are derived for the mean-square fluctuation pressure and pressure correlation in homogeneous turbulence. Applications are made to several idealized flow models involving a mean flow with constant shear, which it is suggested display anisotropy characteristics similar to those in boundary-layer turbulence. The anisotropic pressure correlation, unlike that in the isotropic case, is negative for some values of the separation vector $\mathbf{x}$, and falls off with distance as $|\mathbf{x}|^{-3}$. In general, the principal normalized moments of the pressure correlation tend to be less than those of the velocity correlation. For the idealized cases considered, departure from isotropy results in lower mean-square pressure fluctuations for a given mean kinetic energy of turbulence. *D. W. Dunn* (Baltimore, Md.).

Lyon, Richard H. Propagation of correlation functions in continuous media. J. Acoust. Soc. Amer. **28** (1956), 76–79.

The author considers a situation where a number of elementary events (eddies) $a(\mathbf{r}-\mathbf{y}_i, t)$ combine to form a random function

$$f(\mathbf{r}, t)=\sum_i b_i a(\mathbf{r}-\mathbf{y}_i, t-\tau_i).$$

The vector $\mathbf{r}=\mathbf{y}_i$ specifies the point of origin of the ith eddy, and $t=\tau_i$ the time of origin. The quantities $\mathbf{y}_i$, τ_i are Poisson-distributed, and the b_i are independent equidistributed random numbers. It is shown how the statistics of f may be computed. Applications to cavitation noise and applause are suggested. *E. Reich.*

Curle, N. The influence of solid boundaries upon aerodynamic sound. Proc. Roy. Soc. London. Ser. A. **231** (1955), 505–514.

Lighthill showed [same Proc. **211** (1952), 564–587; MR **13**, 879] that in an infinite homogeneous turbulent medium the sound generated aerodynamically can be regarded as due to a distribution of quadruples of strength,

$$T_{ij}=\varrho v_i v_j+p_{ij}-a_0^2\delta_{ij},$$

where v_i denotes the velocity, p_{ij} the pressure tensor and a_0 the velocity of sound. In this paper Lighthill's result is generalized to the case when the medium is bounded by rigid walls, at which the normal component of the velocity is zero. It is shown that in this case

$$(1)\quad \varrho-\varrho_0=\frac{1}{4\pi a_0^2}\frac{\partial^2}{\partial x_i\partial x_j}\int_V\frac{d\mathbf{y}}{r}T_{ij}(\mathbf{y}, t-r/a_0)$$
$$-\frac{1}{4\pi a_0^2}\frac{\partial}{\partial x_i}\int_S\frac{dS(\mathbf{y})}{r}P_i(\mathbf{y}, t-r/a_0),$$

where the volume integral represents Lighthill's solution and the surface integral represents the effect of the boundaries. The meaning of P_i is that it is the normal component of the pressure tensor at the boundary; also $r=|\mathbf{x}-\mathbf{y}|$. [Note that the values of T_{ij} and P_i are taken at the retarded times $t-r/a_0$.] The additional term in (1) may be regarded as the sound generated by a surface distribution of dipoles. From dimensional arguments the author shows that the dipole term in (1) should predominate at low Mach numbers. *S. Chandrasekhar.*

Berker, Ratip. Sur les équations de compatibilité relatives au mouvement d'un gaz. C. R. Acad. Sci. Paris **242** (1956), 342–344.

This paper gives necessary and sufficient conditions for a vector field $\mathbf{V}$ to be the velocity field of a non-viscous, non-heat-conducting fluid in a given external force field $\mathbf{F}$. The conditions are a system of partial differential equations, which we shall denote by (E); they are, of course, integrability conditions for the equations of motion. Conditions of a similar nature have been given by Hicks, Guenther, and Wasserman [Quart. Appl. Math. **5** (1947), 357–361; MR **9**, 112], and by Munk and Prim [J. Appl. Phys. **19** (1948), 957–958; MR **10**, 215], but only in the special case of gases with a product equation of state, in steady motion, and with $\mathbf{F}=0$. If a vector field V satisfies (E), then one may determine the corresponding equation of state by quadratures; if, however, the equation of state is given a priori, then a further condition must be placed on $\mathbf{V}$ in order that it conform to a flow of fluid having this equation of state. Such a condition is given by the author for a perfect gas and also for a gas whose equation of state is in product form (Prim gas) — gases of these two types therefore admit a kinematic characterization. The author points out that conditions (E) should be useful in determining exact solutions by semi-inverse methods [cf. Neményi, Advances in appl. mech. v. 2, Academic Press, New York, 1951, pp. 123–151; MR **13**, 174], and indeed he states that he has obtained a number of special solutions in this way. Although this note contains only an outline of results, the author says that full details will be published elsewhere.

J. B. Serrin (Minneapolis, Minn.).

Zav'yalov, Yu. S. On some integrals of one-dimensional flow of a gas. Dokl. Akad. Nauk SSSR (N.S.) **103** (1955), 781–782. (Russian)

M. H. Martin [Quart. Appl. Math. **8** (1950), 137–150; MR **14**, 217] has shown that in an unsteady one-dimensional flow the velocity v, time t, and Eulerian coordinate x are of the form

$$v=\Phi_\xi,\ t=\Phi_p,\ x=\int\{(\Phi_\xi\Phi_{\xi p}+1/\varrho)d\xi+\Phi_\xi\Phi_{pp}dp\},$$

where ξ is a Lagrangian coordinate, p pressure, ϱ density, and Φ satisfies

$$(*)\qquad \Phi_{\xi\xi}\Phi_{pp}-\Phi_{\xi p}^2=(1/\varrho)_p.$$

The author remarks that if $(1/\varrho)_p=-f^2(p)$ then $(*)$ has a solution $\Phi=\xi[A\pm\int f dp]+\phi(p)$, where A is constant and ϕ arbitrary. Similarly, if $(1/\varrho)=-f^2(\alpha)/(p+b)$, where $\alpha=(\xi+a)/(p+b)$ and a and b are constants, then $(*)$ has a solution $\Phi=A\pm\int f d\alpha+(p+b)\phi(\alpha)$. *J. H. Giese.*

Dombrovskiĭ, G. A. On integration of the equations of plane parallel steady potential motion of a compressible fluid. Dokl. Akad. Nauk SSSR (N.S.) **103** (1955), 31–34. (Russian)

The velocity potential ϕ and stream function ψ satisfy

$$(1)\qquad \partial\phi/\partial\theta=(\pm K)^{0.5}\partial\psi/\partial s,\ \partial\phi/\partial s=F(\pm K)^{0.5}\partial\psi/\partial\theta,$$

where

$$(2)\quad -(\pm K)^{0.5}dv^{-1}/ds=1/\varrho v,\ d(\varrho v)^{-1}/ds=\mp(\pm K)^{0.5}/v.$$

Here $ve^{i\theta}$ is the velocity, K is a known function of v, ϱ and v are defined as functions of s by (2), and upper (lower) signs refer to sub-(super) sonic flow. The author proposes to approximate K by $[n \tanh m(s-s_0)]^4$ in subsonic flow, and $-K$ by $[n \tan ns]^4$ in supersonic flow, where n, m, s_0 are constants. Then (2) can be integrated explicitly, and the general solution of (1) for these choices of K can be found in terms of several arbitrary functions. The various constants involved can be chosen to yield a third order approximation to the usual adiabat $p=p(\varrho)$ at a value of ϱ corresponding to a given v. *J. Giese.*

Dombrovskiĭ, G. A. Approximate solution of the problem of subsonic flow without circulation about a profile. Dokl. Akad. Nauk SSSR (N.S.) **103** (1955), 777–779. (Russian)

In the subsonic case the approximate general solution of (1) in the preceding paper involves an arbitrary analytic function F of $s-i\theta$. The author proceeds from incompressible circulation-free flow about a given airfoil to describe a method to choose F to obtain a compressible circulation-free flow about a distorted airfoil. *J. Giese.*

Dombrovskiĭ, G. A. Approximate solution of a problem on subsonic flow with circulation about a profile. Dokl. Akad. Nauk SSSR (N.S.) **103** (1955), 985–987. (Russian)

The third paper adapts the method of the second to include circulation about the airfoil. In both cases the flow tends with decreasing Mach number toward incompressible flow with the same circulation. *J. Giese.*

★ **Hayashi, Hiroshi. On the perfect solution of the two-dimensional steady irrotational subsonic compressible fluid flow passing round any convex cylindrical body without circulation.** Proceedings of the Fourth Japan National Congress for Applied Mechanics, 1954, pp. 447–450. Science Council of Japan, Tokyo, 1955.

Ce travail concerne les mouvements plans permanents irrotationnels d'un fluide compressible dont on néglige la viscosité et la conductivité. L'auteur étudie l'équation que vérifie dans le plan de l'hodographe la fonction appellée potentiel de Legendre; il donne de brèves indications sur la façon de construire une solution qui représente l'écoulement subsonique autour d'un cylindre convexe.
H. Cabannes (Marseille).

★ **Imai, Isao. Application of the M^2-expansion method to the subsonic flow of a compressible fluid past a parabolic cylinder.** Proceedings of the First Japan National Congress for Applied Mechanics, 1951, pp. 349–352. Science Council of Japan, Tokyo, 1952.

The technique of the author's earlier paper [Proc. Phys.-Math. Soc. Japan **24** (1942), 120–129; MR **7**, 344] is applied to the problem mentioned in the title. The M^2 terms are found. The author has verified that the results agree with those of Kawaguti [J. Phys. Soc. Japan **6** (1951), 168–174; MR **13**, 180] for elliptic cylinders, in the limiting case of a cylinder of vanishing thickness ratio. *W. R. Sears* (Ithaca, N.Y.).

★ **Simasaki, Tamedi. Application of Poggi's method to the calculation of complex velocity potential for compressible fluid flow.** Proceedings of the First Japan National Congress for Applied Mechanics, 1951, pp. 343–347. Science Council of Japan, Tokyo, 1952.

This is a combination of Imai's method [Proc. Phys.-Math. Soc. Japan (3) **24** (1952), 120–129; MR **7**, 344] and Poggi's [Aerotecnica **12** (1932), 1579–1593; **14** (1934), 532–550] for computing the successive terms in the M^2-expansion method. It is shown that a distribution of spiral vortices (source or sink plus vortex) can be used after removal of certain difficulties caused by singularities, to build up the complementary functions. General results are obtained in the form of contour integrals. Application is made to the flow past a circular cylinder, and Imai's results are confirmed. *W. R. Sears* (Ithaca, N.Y.).

★ **Kawasaki, Tosio. On the subsonic potential flow through arbitrary cascades of airfoils.** Proceedings of the Third Japan National Congress for Applied Mechanics, 1953, pp. 271–274. Science Council of Japan, Tokyo, 1954.

The Kármán-Tsien approximation is used to calculate the flow of a compressible fluid through a lattice of airfoils. This problem was previously treated by Lin [J. Math. Phys. **28** (1949), 117–130; MR **11**, 64], Costello [NACA Rep. no. **978** (1950); MR **12**, 765] and Yeh [J. Aero. Sci. **19** (1952), 630–638]. The present treatment extends the author's method for cascades in incompressible flow [Rept. Transportation Technical Research Inst. Tokyo no. 1 (1951); also J. Jap. Soc. Appl. Mech. **5** (1952), 3–7; MR **14**, 696]. The deformation of the airfoils in the transformation is reduced by successive approximations. Application is made to a typical cascade of low-drag airfoils, and it is found that the turning angle decreases slightly with increasing inlet Mach number although the circulation increases. *W. R. Sears* (Ithaca, N.Y.).

★ **Kasahara, Eiji. Application of the Kármán-Tsien method to the direct potential problem for the compressible flow through the cascade of arbitrary airfoils.** Proceedings of the Third Japan National Congress for Applied Mechanics, 1953, pp. 275–278. Science Council of Japan, Tokyo, 1954.

The subject is the same as that of the paper reviewed above. Again the Kármán-Tsien approximation is used; here the method of Imai [Aeronaut. Res. Inst., Tokyo Imp. Univ., Rep. no. **323** (1946)] for flows with circulation is extended to cascades. The deformation is reduced by successive approximations. An example is worked out, consisting of a staggered cascade of symmetrical profiles at zero incidance. The velocity distributions are compared graphically for the isolated airfoil in incompressible flow and at Mach number 0.6 and for the cascade at the same Mach numbers. In the latter case, both the present theory and the Prandtl-Glauert approximation have been used; the difference between Prandtl-Glauert and Kármán-Tsien results are minute. *W. R. Sears.*

★ **Kasahara, Eiji. Theory of compressible flow through arbitrary cascade of airfoils.** Proceedings of the Fourth Japan National Congress for Applied Mechanics, 1954, pp. 271–274. Science Council of Japan, Tokyo, 1955.

Here the M^2-expansion method is applied to the subject of the title and carried to the first approximation (terms in M^2). The author concludes that this procedure, based on the work of Imai [Aeronaut. Res. Inst., Tokyo Imp. Univ. Rep. no. 275 (1943)], is practical for cascade problems. The results, i.e., velocity distributions for the cascade mentioned in the review directly above are compared with those based on the Kármán-Tsien and Prandtl-Glauert approximations. A similar comparison is made for an isolated airfoil at Mach numbers 0.6 and 0.5. In all cases the present results indicate smaller increments due to compressibility than the other approximations. *W. R. Sears* (Ithaca, N.Y.).

Krasil'ščikova, E. A. Unsteady motion of a wing of infinite aspect ratio. Moskov. Gos. Univ. Uč. Zap. **172** (1954). Meh. **5**, 61–78. (Russian)

The author considers linearized flow about a thin wing governed by the wave equation $a^{-2}\partial^2\phi/\partial t^2-\partial^2\phi/\partial x^2-\partial^2\phi/\partial z^2=0$, where ϕ is the velocity potential, x and z are rectangular coordinates, t is time, and a is the speed of sound. The speed of the wing parallel to the x-axis, the normal velocity, angle of attack, and even the shape of the wing may vary with time. Boundary conditions applied on the x-axis or rather the plane $z=0$ in (x, z, t)-space are: prescribed normal velocity $\partial\phi/\partial z$ on the image of the wing; $\partial\phi/\partial t=0$ on the image of the wake; $\phi=0$ ahead of the wing; and $\partial\phi/\partial z=0$ in undisturbed regions of the fluid. Problems considered include motions starting from rest and accelerated to subsonic or supersonic speeds, supersonic motions with temporary decelerations to subsonic speeds, etc. In all cases she exploits the same

technique repeatedly applied in a long series of her previously reviewed papers, viz.

$$-\pi\phi(t, x, z)=$$

$$\iint_S [\partial\phi/\partial z]_{z=0}[(t-\tau)^2-a^{-2}(x-\xi)^2-a^{-2}z^2]^{-\frac{1}{2}}d\tau d\xi,$$

where S is the intersection of $z=0$ and the sound cone with vertex t, x, z. Where $\partial\phi/\partial z$ is not known apriori in S, it can be obtained from the boundary conditions by solving an Abel integral equation. *J. H. Giese.*

Ginzburg, I. P. Steady flow of a gas from vessels in the presence of friction and local resistances. Vestnik Leningrad. Univ. **10** (1955), no. 5, 55–84. (Russian)

The author presents a one-dimensional treatment of flow from a reservoir into a duct. Changes from stagnation in the reservoir to conditions at the duct inlet are calculated for nonviscous flow. Frictional effects in the duct are taken into account by a term $-\lambda\varrho u^2/8r_0^*$ in the equation of motion, where λ is the frictional coefficient, ϱ density, u velocity, and r_0^* the hydraulic radius of the duct cross-section. Mass flow through the duct is computed as a function of the ratio of exit pressure to reservoir pressure and of a parameter that involves frictional resistance. Several cases are considered: a thermally insulated duct; isothermal flow; flows with heat addition; and flows in ducts with local resistances produced by abrupt changes of cross section. *J. Giese.*

Riabouchinsky, Dimitri. Sur une solution du problème des mouvements presque linéaires non permanents d'une fluide parfait compressible. C. R. Acad. Sci. Paris **241** (1955), 716–718.

The velocity potential function ϕ and two independent stream functions ψ_1 and ψ_2 of an almost uniform flow at speed q_1 parallel to the x-axis may be written

$$\phi=q_1x+\alpha_1 f(x, y, z, t), \quad \psi_1=y\sqrt{q_1}+\alpha_1 f_1(x, y, z, t),$$
$$\psi_2=z\sqrt{q_1}+\alpha_1 f_2(x, y, z, t).$$

To terms of the first order in the small parameter α_1, f, f_1, and f_2 satisfy

$$(*)\quad (1-q_1^2c^{-2})\partial^2 f/\partial x^2+\partial^2 f/\partial y^2+\partial^2 f/\partial z^2-$$
$$2q_1c^{-2}\partial_f^2/\partial x\partial t-c^{-2}\partial^2 f/\partial t^2=0.$$

The author considers solutions in the form of series of terms $F_\nu(\xi_\nu)-F_\nu(\overline{\xi_\nu})$ for f, and $F_\nu(\xi_\nu)+F_\nu(\overline{\xi_\nu})$ for f_1 and f_2, where $\xi_\nu(\overline{\xi_\nu})=\gamma_\nu x-\beta_\nu t+(-)\varkappa a(b_{1\nu}y+b_{2\nu}z)$ where γ_ν, β_ν, $\varkappa a$, $b_{1\nu}$, $b_{2\nu}$ are constants. To the same order of approximation the independent variables in (*) may be replaced by ϕ/q_1, $\psi_1/\sqrt{q_1}$, and $\psi_2/\sqrt{q_1}$. *J. Giese* (Aberdeen, Md.).

Riabouchinsky, Dimitri. Condition d'existence des solutions de l'équation régissant les mouvements presque linéaires non permanents d'un fluide parfait compressible. C. R. Acad. Sci. Paris **241** (1955), 1012–1014.

In the second paper the author proposes, in effect, that the non-vanishing of the first-order approximation to the Jacobian

$$\partial(f, f_1, f_2)/\partial(\phi, \psi_1, \psi_2)=1-\alpha_1(\partial f/\partial\phi+\partial f_1/\partial\psi_1+\partial f_2/\partial\psi_2)$$

can be used to estimate the maximum value of α_1 for which his linearization is valid. He also exhibits a particular choice of ϕ and ψ_1 which is said to correspond to unsteady flow in a jet issuing from a vibrating nozzle's exit. *J. Giese* (Aberdeen, Md.).

Kryučin, A. F. Uniqueness theorem for the solution of the problem of transonic gas flow about a wedge-shaped profile. Prikl. Mat. Meh. **19** (1955), 639–640. (Russian)

F. I. Frankl [Izv. Akad. Nauk SSSR. Ser. Mat. **9** (1945), 121–143; MR **7**, 496] has shown that if a solution of the boundary-value problem for steady irrotational flow about a finite wedge exists it is unique. With the aid of a lemma of F. Tricomi's [Atti Accad. Naz. Lincei. Mem. Cl. Sci. Fis. Mat. Nat. (5) **14** (1923), 133–247] the author proves uniqueness for the corresponding problem for the usual simplified equations of transonic flow in a distorted hodograph plane. *J. Giese* (Havre de Grace, Md.).

Kališevič, I. Z. Solution of boundary problems for supersonic motion of a gas without density discontinuities. Dokl. Akad. Nauk SSSR (N.S.) **102** (1955), 1085–1088. (Russian)

The author has previously developed an approximation of unspecified accuracy for the stream function of steady plane rotational flow which involves three arbitrary functions and several arbitrary constants. [same Dokl. (N.S.) **99** (1954), 37–40; MR **16**, 641]. Now he has shown how to determine these functions in order to solve four standard problems with boundary data on (1) a non-characteristic curve; (2) intersecting characteristics; (3) a characteristic and a jet boundary; (4) a characteristic and a rigid wall. *J. Giese* (Aberdeen, Md.).

Bulah, B. M. On the theory of nonlinear conical flows. Prikl. Mat. Meh. **19** (1955), 393–409. (Russian)

The author considers irrotational conical flows under the assumption that any shocks that occur are so weak that vorticity and variations in entropy are negligible. A conical flow field adjacent to a region of supersonic flow must be a simple wave, in which one family of characteristic surfaces is composed of planes. There is an extensive discussion of the relation of the envelop of these planes to a parabolic surface of the governing partial differential equation. The results are applied to an attempt to determine the qualitative nature of a symmetrical conical flow at moderate angle of attack over a plane wing with swept forward supersonic leading and trailing edges. On the upper (lower) surfaces there appear the expected expansions (plane shocks) at the leading edges, followed by regions of uniform flow. Efforts to supplement these by the required simple wave patterns lead to contradictions which force the author to conclude that additional shocks occur in the disturbed flows on each side of the wing. Within these shocks he formulates boundary value problems of elliptic type which he has solved by relaxation methods in a particular case.

J. Giese (Aberdeen, Md.).

Tsien, S. H. The supersonic conical wing of minimum drag. J. Aero. Sci. **22** (1955), 805–817, 843.

The lift distribution of conical wings with subsonic leading edges is expressed in series form, where the individual terms represent (1) uniform lift, (2) uniform downwash (flat-plate wing), and (3) an infinite number of distributions that vanish at the leading edge. This is an extended form of a series used by Baldwin [NACA Tech. Note no. **1816** (1949)]. The condition of minimum drag-due-to-lift is then applied. If the full theoretical value of leading-edge suction is present, the minimum drag is practically that of a flat-plate wing of the same planform at the same lift coefficient, except when the leading edge

is nearly sonic. If, on the contrary, it is assumed that no leading-edge suction is obtained — a conservative approximation that is sometimes made for wings with rather sharp edges — the drag of a flat wing is much greater but a minimum-drag cambered wing is found whose drag is again practically the same as before. Thus, camber is important in drag reduction if leading-edge suction is not complete. The camber and lift distributions for these cases are worked out.

In an appendix, an investigation of the effects of parabolic chordwise camber is made. It is found that this camber offers considerable improvement in drag compared to the conical family, at least when edge-suction is absent and the sweepback is large. *W. R. Sears* (Ithaca, N.Y.).

Legendre, Robert. Ecoulement supersonique autour d'un corps élancé. C. R. Acad. Sci. Paris **242** (1956), 730–732.

Frankl', F. I. Energy equations for motion of a fluid with suspended sediment. Dokl. Akad. Nauk SSSR (N.S.) **102** (1955), 903–906; erratum **106** (1956), 382. (Russian)

Dans un travail précédent [mêmes Dokl. (N.S.) **92** (1953), 247–250; MR **15**, 578] l'auteur a déjà déduit les équations de continuité ainsi que les équations du mouvement pour les moyennes sans introduire de nouvelles hypothèses en utilisant seulement les lois générales de mécanique des milieux continus. Dans ce travail l'auteur déduit d'une manière analogue les équations de l'énergie. On trouve de cette façon six équations: une paire d'équations des énergies cinétiques pour les mouvements moyens de chaque phase (solide et liquide), une paire pour l'énergie cinétique moyenne des pulsations de chaque phase; et deux équations pour l'énergie calorifique moyenne de chaque phase. Toutes ces équations ont été obtenues en supposant le liquide incompressible et les particules solides indéformables. *M. Kiveliovitch* (Paris).

Andreev, N. N. On some quantities of second order in acoustics. Akust. Ž. **1** (1955), 3–11. (Russian)

The author reviews recent corrections to expressions for the average density of energy in acoustic fields and its relation to intensity and radiation pressure. First- and second-order effects are involved and the need for distinguishing accurately between the Lagrangian and Eulerian points of view is stressed. Initial and boundary conditions must be included to arrive at a correct explanation. The author indicates that future studies of important phenomena in second-order acoustic fields will require inclusion of viscosity and heat conduction.
W. W. Soroka (Berkeley, Calif.).

Ginzburg, V. L. On a general connection between absorption and dispersion of sound waves. Akust. Ž. **1** (1955), 31–39. (Russian)

The author cites a universal relation between attenuation and dispersion of electromagnetic waves [D. E. Kerr (ed.), Propagation of short radio waves, McGraw-Hill, New York, 1951, Sec. 8.1]. He transfers the electromagnetic relations to the acoustic case, pointing out the similarity of the complex sound speed to the complex dielectric constant, both being functions of frequency. From the derived relations one may evaluate attenuation if dispersion in known, and vice versa. Differences between the acoustic and electromagnetic cases are discussed.
W. W. Soroka (Berkeley, Calif.).

Lysanov, Yu. P. Scattering of sound from a plane nonhomogeneous surface with periodically varying acoustic conductance. Akust. Ž. **1** (1955), 58–69. (Russian)

Acoustic conductance of a plane surface, independent of angle of incidence, is expressed by $\eta(x, y)=\eta_0[1+hW(x, y)]$, with W periodic and of zero mean value. The sound pressure field resulting from incidence on, and reflection and scattering by, the surface is expressed in a power series in h, for whose mth coefficient a recurrent integral relation is developed. Integration is accomplished for special forms of W, such as $\cos qx$ for which the reflected field consists of mirror-type reflections plus scattered side spectra of various orders on both sides of the mirror reflections, and $\exp(iqx)$ for which side spectra occur only on one side of the mirror reflections. Convergence of each series is discussed. *W. W. Soroka.*

Blokhintsev, D. I. Acoustics of a nonhomogeneous moving medium. NACA Tech. Memo. no. **1399** (1956), iv+194 pp.

Translation of Blohincev's Akustika neadnorodnoĭ dvižuščeĭcya sredy [Gostehizdat, Moscow, 1946].

See also: Chen, p. 624; Putnam, p. 637; Binnie and Miller, p. 672; Fil'čakov, p. 673; Graffi, p. 688; Piddington, p. 691.

Elasticity, Plasticity

Aržanyh, I. S. On a connection of a bi-wave field with the dynamical theory of elasticity. Dokl. Akad. Nauk SSSR (N.S.) **104** (1955), 520–523. (Russian)

The author shows that there is a direct connection between the integration of the biwave equation

$$(*)\qquad \left(\nabla^2-\frac{1}{\alpha}\frac{\partial^2}{\partial t^2}\right)\left(\nabla^2-\frac{1}{\beta}\frac{\partial^2}{\partial t^2}\right)\varphi=f(x_1, x_2, x_3, t),$$

and the integration of the dynamical equations of three-dimensional elasticity

$$(**)\qquad \alpha \operatorname{grad}\operatorname{div} u-\beta \operatorname{rot}\operatorname{rot} u-\frac{\partial^2 u}{\partial t^2}=F(x_1, x_2, x_3, t);$$

and also between the Laplace transform of the equation

$$\prod_{i=1}^{2}\left(\nabla^2-k_i^2-\frac{1}{c_i^2}\frac{\partial^2}{\partial t^2}\right)\varphi=f,$$

$k_i=2\pi m_i c_i/h$, where h is Planck's constant, and that of the Laplace transform

$$\alpha \operatorname{grad}\operatorname{div} v-\beta \operatorname{rot}\operatorname{rot} v-\eta^2 v=G(x_1, x_2, x_3, \eta),$$

of (**). The properties of the potentials of simple and double layers corresponding to equation (*) are investigated. *J. B. Diaz* (College Park, Md.).

Teleman, Silviu. La méthode de la projection orthogonale et les deux premiers problèmes de la théorie de l'élasticité. Acad. Repub. Pop. Romîne. Bul. Şti. Secţ. Şti. Mat. Fiz. **7** (1955), 105–125. (Romanian. Russian and French summaries)

Dans cet article, on applique la méthode de la projection orthogonale pour résoudre deux problèmes posés par la théorie de l'élasticité: 1) déterminer le déplacement à l'intérieur, lorsqu'on le connait sur la frontière, et 2) déterminer le déplacement à l'intérieur, lorsqu'on connait la pression sur la frontière d'un domaine rempli d'un

milieu élastique, homogène et isotrope, sur lequel n'agissent pas des forces de volume. (Resumé de l'auteur.)
J. Deny (Princeton, N.J.).

Golecki, Józef. Axially symmetrical problems concerning bodies bounded by spherical surfaces. Arch. Mech. Stos. 7 (1955), 201–220. (Polish. Russian and English summaries)

A method is described for the solution of problems with polar symmetrical boundary conditions in terms of stress, based on the expansion of the stress components in series of Legendre polynomials, which is an extension of Thompson's solution for a spherical surface with prescribed displacements. Solutions are presented for various states of stress around a spherical cavity, as well as for a thick-walled multilayer spherical shell. *A. M. Freudenthal.*

Ramakanth, J. Finite torsion of aeolotropic and composite cylinders. I. Z. Angew. Math. Mech. **35** (1955), 453–459. (German, French and Russian summaries)

Using a special linear relation between stress and strain (defined with reference to the deformed state), which is stated to be applicable to hexagonally aeolotropic bodies, the author solves the problem of finite torsion of a long circular cylindrical shell. The torsional couple is found and evaluated numerically using values of the elastic constants for Beryl. The corresponding results for an isotropic cylinder are also given. [It is not clear to the reviewer that the stress-strain relations used are correct for finite deformations of a hexagonal aeolotropic body. Hexagonal aeolotropy is a property of the undeformed body and the author uses strain referred to the deformed body. Also, the elastic constants quoted for Beryl are apparently those found from experiments using classical infinitesimal elasticity theory. No discussion is given about the validity of using these values in a finite-strain theory.] *A. E. Green* (Providence, R.I.).

Saito, Hideo. Torsion of a circular shaft press-fitted with a disc. Z. Angew. Math. Phys. 6 (1955), 498–503.

This is a short paper dealing with a circular shaft which is press-fitted with a disc and which is under torsion, the shaft at a great distance from the disc being in a state of pure shear having a moment M. It is assumed that there is no slip at the surface of contact between the disc and the shaft and the components of stress and displacement are expressed as infinite series in terms of Bessel functions of the first and second kind. The unknown constants in these infinite series are obtained in special cases by a method of successive approximations. Numerical calculations have been carried out for particular values of the radii of the shaft and disc and the thickness of the disc. *R. M. Morris* (Cardiff).

Vorovič, I. I. On some direct methods in the nonlinear theory of sloping shells. Dokl. Akad. Nauk SSSR (N.S.) **105** (1955), 42–45. (Russian)

The 3 non-linear equations governing the large deflections of thin shells can be reduced to a single integro-differential equation in the normal displacement if the existence of Green's tensor for the plane (linear) problem and of Green's function $\mathcal{G}(P, Q)$ for the biharmonic equation is assumed for the region considered. If $\mathcal{G}(P, Q)$ can be written in the form

$$\mathcal{G}(P, G)=\sum \phi_k(P)\phi_k(Q)$$

the method of Bubnov-Galerkin is applicable. Existence theorems are stated. Their proofs apparently follow from results of Krasnosel'skiĭ [same Dokl. (N.S.) 73 (1950), 1121–1124; MR 12, 187]. *R. C. T. Smith.*

Ševlyakov, Yu. A. Stress concentration in a cylindrical shell with a circular cut-out on the lateral surface. Dopovidi Akad. Nauk Ukrain. RSR **1955**, 123–125. (Ukrainian. Russian summary)

The author considers a cylindrical shell which at one end is strengthened by a rigid ring and has a circular hole cut out in the lateral wall. The shell is subjected to a distributed constant load acting on the inner surface (which may be caused for example by a tightly driven bolt). The author writes down ready formulas for stresses and displacements which were derived by the method of complex functions by A. I. Lur'e [Statics of thin-walled elastic shells, OGIZ, Moscow-Leningrad, 1947; MR 12, 301]. These formulas valid in general are transformed by the author to satisfy his particular boundary conditions.
T. Leser (Aberdeen, Md.).

Szelągowski, Franciszek. A couple acting on the periphery of a circular hole in an infinite plate. Arch. Mech. Stos. 7 (1955), 337–344. (Polish. Russian and English summaries)

The author considers an infinite plate with a circular hole and a couple represented by two forces S acting at two opposite points on the circumference of the hole. Resolving the force S into a normal force Q and a tangential force P, the problem can be broken into two problems; one of finding the stresses due to Q, and one of finding the stresses due to P. The total stress is the sum of the stress due to Q and the stress due to P.

The stresses due to normal forces Q were found previously by the author [same Arch. 6 (1954), 365–388; MR 16, 647]. Thus, in this work the author finds only the stresses due to the tangential forces P. His procedure can be described as follows: the concentrated tangential force P is replaced by a distributed tangential load acting on the part 2α of the circumference. The plane elasticity equations involving complex functions lead to a Poisson integral which is solved in a closed form. Boundary conditions determine the constant of integration. Finally the stresses due to concentrated loads are found as limits when α approaches zero. *T. Leser* (Aberdeen, Md.).

Koiter, W. T.; and Alblas, J. B. On the bending of cantilever, rectangular plates. I, II, III. Nederl. Akad. Wetensch. Proc. Ser. B. 57 (1954), 250–258, 259–269, 549–557.

In part I, the authors deal with the stress distribution in a thin elastic plate occupying a quarter plane, with one edge clamped and the other unsupported but subject to prescribed bending moments. Classical thin-plate theory is postulated, and a formal solution is obtained by using Fourier transforms to deduce certain integral equations which are then solved by use of Mellin transforms. In part II, the formal solution of I is first verified. The stress resultants are then evaluated analytically and numerically for a piecewise constant edge-moment distribution and the significance of a singularity in the stress distribution at the corner is discussed. Part III deals with the corresponding problem for a semi-infinite strip. The problem is reduced, by use of preceding results, to that of solving a certain infinite set of linear algebraic equations. It is stated that this set is more amenable to

iterative methods of approximate solution than the set which corresponds to an alternative treatment by a more conventional approach, which is also given, although convergence of the iteration is not established analytically. A discussion of corresponding numerical results is reserved for a part IV to follow. *F. B. Hildebrand.*

Nowacki, Witold. Some problems of dynamics and stability of a rectangular plate with discontinuous boundary conditions. Arch. Mech. Stos. 7 (1955), 266–284. (Polish. Russian and English summaries)

The author considers a rectangular plate compressed laterally by a load q distributed on two opposite edges and carrying also a normal distributed periodic load, $p(x, y)\sin\omega t$. The boundary conditions are as follows: three edges are freely supported, a part of the fourth edge is freely supported or free, the remaining part is built-in. The total deflection of the plate is a sum of a deflection w_0 of a freely supported plate with a normal load $p\sin\omega t$, and a deflection w_1 of a freely supported plate with a moment $M\sin\omega t$ distributed along the built-in edge. The Green function in this case is the equation of the deflection surface caused by a concentrated unit moment $1\cdot\sin\omega t$ acting at one point at the built-in edge. The procedure leads to a Fredholm integral equation of the first kind where the unknown function is the moment M. The author evaluates the Green function which in turn permits him to find M from the Fredholm integral equation. An approximate solution of the last equation where the integral is replaced by a sum leads to a system of homogeneous equations. The natural vibrations and the critical load q can be found from the determinant of this system.

The general procedure is illustrated on examples of: (i) a square plate a by a where one edge is built-in along the segment $a/2$, the remaining segment $a/2$ and the other edges are freely supported, on rectangular plates where, (ii) part of one edge is built-in, the remaining part is free, and the other three edges are freely supported, (iii) part of one edge is free, the remaining part and the other three edges are freely supported, (iv) freely supported all edges with a slit along one axis of symmetry of the plate.

T. Leser (Aberdeen, Md.).

Nowiński, Jerzy. Certain characteristic points of cross-sections of thin walled tubes. Rozprawy Mat. 7 (1954), 52 pp. (Polish. Russian and English summaries)

In this short treatise the author summarizes his theory of thin-walled beams of constant cross-section which he developped in a series of previous works. In the earlier publications the author not only developped his theory of thin-walled beams but also showed many erroneous notions in previous theories. He redefined centers and axes of shear and twist, showed that the axis of twist is not a straight line, that its position depends on supports and loads and that the center of twist does not coincide with the center of shear.

In this paper the author derives briefly the basic formulas, applies them to a beam of an arbitrary constant cross-section loaded by a concentrated force and shows a procedure for finding the center of shear. This is followed by a more concrete example of a beam of a trapezoidal cross-section and a numerical example of a beam of rectangular cross-section. In order to simplify computations the author makes an unrealistic assumption that the beam carries only shearing stresses all other stresses being absent, and claims that this dies not lead to a significant error. *T. Leser* (Aberdeen, Md.).

Jones, E. E. The flexure of a non-uniform beam. Pacific J. Math. 5 (1955), 799–806.

L'A., après avoir rappelé l'usage fait par divers auteurs de la transformation de Laplace pour la résolution du problème du titre, propose un procédé pour un calcul direct convenablement approximé, consistant essentiellement en la division de la longueur de la poutre en segments suffisamment courts pour permettre de considérer sur chacun d'eux simplement linéaire la variation de la charge et de la rigidité et par conséquence immédiate l'intégration. *B. Levi* (Rosario).

Lovass-Nagy, V. Mathematical investigation of the stability against lateral buckling of a beam freely suspended at both ends. Acta Tech. Acad. Sci. Hungar. 13 (1955), 281–297. (Russian, French, and German summaries)

Verf. untersucht die elastische Stabilität gegen seitliches Ausknicken eines in je einem Schwerpunkt seiner beiden Endquerschnitte frei gelagerten Stabes mit zwei zu einander senkrechten Symmetrie-Ebenen und zwar für den Fall, dass die Lagerungspunkte in eine der Hauptträgheitsachsen der Endquerschnitte fallen, und die Endquerschnitte in ihren Hauptträgheitsebenen frei drehbar sind. Es werden die Bedingungen für den allgemeinen Belastungsfall aufgestellt, bei dem der Stab durch eine vertikale und nach einer beliebigen Funktion verteilte Belastung beansprucht wird, die längs der Schwerpunktsachse des Stabes wirkt. Eine Näherungslösung des Problems, die den Randbedingungen genügt, wird durch eine Modifikation der Methode Ritz-Galerkin mit Hilfe der Matrizenrechnung gefunden. Schliesslich wird jene kritische Exzentrizität der Lagerungspunkte der Endquerschnitte ermittelt, bei welcher der Stab aus seiner stabilen Gleichgewichtslage seitlich nicht ausknickt, bzw. bei welcher der Stab in seiner stabilen Gleichgewichtslage ausgebogen bleibt. Die Arbeit ist sprachlich so mangelhaft geschrieben, dass sie an manchen Stellen schwer zu verstehen ist.

R. Gran Olsson (Trondheim).

Lakshmana Rao, S. K. On the vibrations of triangular membranes. J. Indian Inst. Sci. Sect. B. 38 (1956), 1–3.

Piszczek, Kazimierz. Longitudinal and transversal vibrations of a rod subjected to axially pulsating force, taking nonlinear members into consideration. I. Arch Mech. Stos. 7 (1955), 345–362. (Polish. Russian and English summaries)

The author investigates a problem of dynamic stability of a straight vertical rod of constant cross-section hinged at the upper and lower ends and under an axial load of the form $P=P_0+P_1\cos\omega t$. His main considerations are longitudinal vibrations but the influence of the transverse deflection is taken into account. The physical nonlinearity of the system consists of the non-linear stress-strain relation. The stress is not a linear function of the strain but is a third-degree polynomial of the strain.

The system of equations controlling vibrations of such a rod is obtained from Hamilton's principle. To solve this system the author introduces the following simplifying assumptions: (i) the model has only one degree of freedom for longitudinal and transverse vibrations, and (ii) the longitudinal vibrations do not influence the transverse

vibrations. Using these assumptions and applying Galerkin's method he reduces the system to the well known Mathieu differential equation whose solutions are sine and cosine type Mathieu functions. The author makes a detailed analysis of the eigenvalues (related to the frequencies of the forced vibrations) for which the system is stable or unstable. *T. Leser* (Aberdeen, Md.).

Karas, K. Eigenschwingungen von Saiten mit elastisch befestigten Enden. Österreich. Ing.-Arch. **9** (1955), 352–388.

Vibration modes of a stretched string with elastically supported ends are studied, principally in the case in which the string is uniform and without bending stiffness. This problem is used to compare the methods of Ritz and of Grammel for approximating the oscillational frequencies, illustrated by extensive numerical calculations.

Three non-uniform cases are then treated. In a), the density per unit length of the bar jumps at an interior point from a constant value on one side to a different constant value on the other side; the solution involves circular functions. In b), the density per unit length varies linearly; the solution involves Bessel functions of order $\pm 1/3$. In c), the density par unit length varies quadratically; the solution involves Whittaker functions or, in case the quadratic variation is a perfect square, Bessel functions of order $\pm 1/4$.

Finally a uniform string is treated with bending stiffness considered. *E. Pinney* (Berkeley, Calif.).

Marhasev, G. S. Head waves in elastic media with plane boundaries. Prikl. Mat. Meh. **19** (1955), 165–178. (Russian)

Two semi-infinite media are assumed to be in welded contact. A point source is located in the medium of lower velocity (a_1) of wave propagation. This problem, solved before by different methods [e.g., M. Muskat, Physics **4** (1933), 14–28; or L. Cagniard, Réflexion et réfraction des ondes séismiques, Gauthier-Villars, Paris, 1939] is now attacked by the method of Smirnov and Sobolev [Trudy Seĭsmol. Inst. Akad. Nauk SSSR no. **20** (1932)]. The author considers the two-dimensional problem as well as the axi-symmetrical case and follows Zaĭcev and Zvolinskiĭ [Izv. Akad. Nauk SSSR. Ser. Geograf. Geofiz. **15** (1951), 20–39; Izv. Akad. Nauk SSSR. Ser. Geofiz. **1951**, no. 5, 40–50; MR **12**, 650; **13**, 512] in choice of the source type, namely,

$$\varphi_0(r, z, t)=\begin{cases}1-a_1t(r^2+z_0^2)^{-\frac{1}{2}} \\ 0\end{cases} \quad \text{for } r^2+z_0^2 \lesseqgtr a_1^2t^2.$$

General conclusions concerning the wave types and wave fronts coincide with those of Gagniard. *W. Jardetzky* (New York, N.Y.).

Litwiniszyn, Jerzy. On a type of heterogeneity of elastic bodies. Arch. Mech. Stos. **7** (1955), 301–314. (Polish. Russian and English summaries)

The effect is discussed of the heterogeneity of a continuous medium due to the spatial and temporary variation of the relevant physical constants associated either with density variations resulting from deformation or flow of the medium, or with imposed external nonmechanical effects, such as a magnetic field or chemical reaction- or concentration-gradients. The one-dimensional equation of motion is established formaly. *A. M. Freudenthal* (New York, N.Y.).

Prager, William. Théorie des plaques plastiques. Bull. Tech. Suisse Romande **81** (1955), 85–90.

The yield condition and flow rule are derived for a rigid plastic plate in terms of the radial and circumferential moments and the deflection rate. Examples of finding the load carrying capacity are given. An illustration is given for determining a lower bound on the minimum weight of a plate. Large deflections are discussed briefly and a comparison of theory and experiment are made. An example is given of a dynamic load problem in which a load greater than the collapse load is applied for a very short time.

The paper contains little that has not been previously presented by the author and his associates at Brown University [Hopkins and Prager, J. Mech. Phys. Solids **2**, (1953), 1–13; Z. Angew. Math. Phys. **5**, (1954), 317–330; MR **15**, 270; **16**, 648; Onat and Prager, Nederl. Acad. Wetensch. Proc. Ser. B. **57** (1954), 534–548; MR **16**, 649; Onat and Haythornwaite, J. Appl. Mech. **23** (1956), 49–55; J. Aero. Sci. **22** (1955), 867–869]. However, in the words of the author, "A survey··· is offered in the present paper as a convenient introduction to the study of of the work presented in greater detail in a number of recent papers and reports, some of which are not readily accessible." *P. G. Hodge, Jr.* (Brooklyn, N.Y.).

Shield, R. T. On the plastic flow of metals under conditions of axial symmetry. Proc. Roy. Soc. London. Ser. A. **233** (1955), 267–287.

The axially symmetric problem is described by the four stress components σ_r, σ_θ, σ_z, and τ_{rz} and two displacement components. For a rigid-perfectly plastic material satisfying Tresca's yield condition, the stress variables are related by two equilibrium equations and the condition $\sigma_1-\sigma_3=2k$, where σ_1, σ_2, σ_3 are the principal stresses in descending order. Haar and Kármán [Nachr. Ges. Wiss. Göttingen. Math.-Phys. Kl. **1909**, 204–218] proposed that such problems be solved with the additional requirement that two of the principal stresses be equal, thus obtaining four stress equations for the four stresses.

Later investigators pointed out that the Haar-Kármán assumption had no validity with relation to the then more widely used Mises yield condition, and the subject languished. In the present paper, the problem is examined from the viewpoint of the Tresca yield condition and the associated flow law. It is found that the Haar-Kármán assumption is in fact valid for a wide class of axially symmetric problems, provided that it is associated with the proper flow law for a singular point of the Tresca yield condition. Complete solutions are found for the incipient necking of a circular cylindrical rod under tension, and for the indentation of a semi-infinite plastic region by a rigid circular punch. *P. G. Hodge.*

Bilby, B. A.; Bullough, R.; and Smith, E. Continuous distributions of dislocations: a new application of the methods of non-Riemannian geometry. Proc. Roy. Soc. London. Ser. A. **231** (1955), 263–273.

Bei der analytischen Beschreibung eines Kristalles, der eine ganz beliebige Verteilung von Versetzungslinien enthält, ist es nach J. F. Nye [Acta Metallurgica **1** (1953), 153–162] zweckmässig, diese Versetzungen in einem gewissen Sinne zu verschmieren, also unendlich viele Versetzungen anzunehmen, deren Stärke jedoch zu Null geht. (Selbstverständlich ist das bloss eine mathematische Abstraktion, weil ja die Materie atomistisch aufgebaut ist.) Nach dieser Betrachtungsweise ist jedoch der Kristall

nirgends perfekt und es entstehen eben deshalb bei der Definition der Burgersschen Ringe und im Zusammenhang damit mit dem Begriff des Versetzungstensors gewisse Schwierigkeiten. Die Verfasser umgehen diese Schwierigkeit damit, dass sie das Gitter in seinem (deformierten) Endzustande mit Hilfe einer nichtriemannschen Geometrie, ganz analog zur einheitlichen Feldtheorie von Einstein, also unter Aufrechterhaltung des Begriffes des Fernparallelismus, beschreiben. Der Tensor der die lokale Dichte des Versetzungstensors beschreibt, wird dann mit dem Torsionstensor dieser Geometrie in Zusammenhang gebracht. Weiter wird darauf verwiesen, dass der affine Krümmungstensor

$$L^{\alpha}_{\beta\gamma\delta}=\frac{\partial L^{\alpha}_{\beta\delta}}{\partial c^{\gamma}}-\frac{\partial L^{\alpha}_{\beta\gamma}}{\partial c^{\delta}}+L^{\sigma}_{\beta\delta}L^{\alpha}_{\sigma\gamma}-L^{\sigma}_{\beta\gamma}L^{\alpha}_{\sigma\delta}$$

auch hier identisch verschwindet, so dass die endliche parallele Verschiebung von Gittervektoren unabhängig vom Wege ist, was ja eine notwendige Bedingung der einheitlichen Definition des Gitters ist. *T. Neugebauer.*

See also: Albrecht, p. 628; Pleijel, p. 628; Mal'nik, p. 629; Galustyan, p. 630; Slobodyanskiĭ, p. 648.

MATHEMATICAL PHYSICS

Graffi, Dario. Alcuni problemi non lineari della fisica matematica. Univ. e Politec. Torino. Rend. Sem. Mat. **14** (1954–55), 75–86.

Expository paper dealing with certain problems of electro-magnetic theory, fluid mechanics, magneto-hydrodynamics and heat conduction.

See also: Farinelli and Gamba, p. 571; Jeffreys and Jeffreys, p. 590; Leti, p. 644; Arcidiacono, p. 644; Rahman, p. 671.

Optics, Electromagnetic Theory, Circuits

Laudet, Michel. Optique électronique des systèmes cylindriques présentant un plan de symétrie. II. Les aberrations. J. Phys. Radium (8) **16** (1955), 908–916.

Dans un premier article [même J. (8) **16** (1955), 118–124; MR **17**, 107], M. Laudet a étudié les trajectoires des particules dans le cas de l'approximation du premier ordre (champ électrique, champ magnétique, relativité, etc.). Dans ce second article, il établit les formules concernant les diverses aberrations suivant le schéma méthodique d'approximations successives proposé par E. Durand [Rev. Opt. **33** (1954), 617–629; MR **16**, 652].

L'auteur calcule d'abord les aberrations chromatiques. La considération d'un diaphragme linéaire le conduit à définir les aberrations chromatiques de rotation, de translation, d'ouverture et l'aberration chromatique longitudinale. Il détermine ensuite, à partir des équations de Lagrange, les aberrations du troisième ordre. La méthode consiste à développer en série l'expression du Lagrangien: $F=F_0+F_2+F_4\cdots$, et à se limiter aux termes du quatrième ordre. Le calcul des coefficients d'aberrations est fait par la méthode classique donnant la solution d'une équation différentielle avec second membre quand on connaît la solution générale de l'équation sans second membre. Toujours en considérant un diaphragme linéaire, il étudie quelques courbes d'aberrations et détermine en particulier leur courbure. Enfin, il étudie plus spécialement le cas des systèmes électrostatiques purs pour lesquels il n'y a pas de rotation de l'image. *E. Durand.*

Glaser, W. und Schiske, P. Strenge Durchrechnung einer typischen elektrostatischen Einzellinse. Optik **11** (1954), 422–443, 445–467.

In this paper the authors have investigated the electron-optical properties of a bell-shaped electrostatic axial potential distribution in superposition with a magnetic-field distribution of the same form. The paraxial-ray equation for such a field system can be integrated exactly, the solution being expressed by a combination of trigonometric and first Legendre elliptic functions with arguments $d\cot^{-1}z$, z being the axial coordinate and d is the distance from a point z when the potential is reduced to half its maximum value. The focusing properties of the lens are analyzed in detail from the behavior of the elliptic functions. The authors have derived expressions for the focal points, magnification, cardinal planes and chromatic aberration for a parallel incident beam of electrons. By extending the potential distribution to three dimensions by means of the Laplace-Whittaker method, the spatial potential distribution resembles that of the classical problem of two centers with an imaginary mass and where the location of one of the centers is the conjugate complex of the other. By introducing oblate spheroidal coordinates and limiting the case to the meridian plane, the authors have obtained the paths of the electrons from the solution of the Hamilton-Jacobi equation. For the case of an incident axial parallel beam the authors describe various effects depending on the incident velocity of the electrons, i.e., filtering (electron), telescopic behavior of the lens and velocity separation. The paper concludes with an analysis of the lens when the inner potential is positive and a brief discussion of non-symmetric bell-shaped field distribution. *N. Chako.*

Glaser, Walter; und Braun, Günther. Zur wellenmechanischen Theorie der elektronenoptischen Abbildung. II. Acta Phys. Austriaca **9** (1955), 267–296.

The general theory presented in Part I [same Acta **9** (1954), 41–74; MR **16**, 651] is applied to the calculation of the wave-mechanical intensity distribution in the paraxial region of a rotationally summetrical electromagnetic field. An explicit integral representation is derived for the wave function $\psi(z, x, y)$, and its behavior is investigated in the Fraunhofer plane and its vicinity. For points sufficiently far removed from the Fraunhofer plane, ψ is evaluated both for geometrical and wave optics. Explicit expressions in the form of asymptotic series are given for ψ at points in the illuminated space, the dark space, and the shadow outline. Conclusions are given as to the nature of the image in geometrical optics. In the case of wave optics, ψ is expressed by a series of integrals over the boundary of the diaphragm, and it is indicated how such integrals can be evaluated by the method of stationary phase. Applications are given to the case of the infinite slit, the straight edge, the polygon, and the circular aperture respectively. *J. E. Rosenthal.*

Müller, Johannes. Untersuchungen über Elektronenströmungen. Z. Angew. Math. Physik **5** (1954), 203–232.

In this paper the author treats the problem of electron

streaming under the influence of a weak external fluctuating electric field superposed on a strong static field. By neglecting the effect of the weak fluctuating magnetic field on the stream motion of the electrons, an equation for the transport power is derived which contains an inductive term resulting from the small velocity fluctuations (voltage modulation) of the stream. Using matrix methods, the properties of the stream flow and the conditions for the conservation of charge and energy are derived by assuming small fluctuations about the static state, thus necessarily limiting the treatment to linear phenomena (effect of collisions are excluded). The author considers two cases: (a) streams with a constant spectrum (variation of current with respect to the initial velocity of the electrons remains constant along the path); (b) the spectrum is of exponential form (Maxwellian). Case (b) corresponds to the problem of thermal effects; i.e. thermal noise in tubes. *N. Chako* (New York, N.Y.).

Hagen, G. B. Über die Konstruktion von Elektronenbahnen in Potentialfeldern. Ann. Physik (6) 13 (1953), 257–284.

In this paper the author examines the accuracy of two methods which are used in constructing electron paths in electrostatic fields. i.e., (a) by calculating the curvature radius of the path (arc construction) from the relation between the velocity of the electrons and the electrostatic potential, and (b) by employing Snell's law for the incident and refracted ray between successive equipotentials. By considering a potential distribution of the form, $\Phi=k\varphi$ (r, φ are cylindrical coordinates in the plane), the author has calculated the deviation angle between the incident ray at one potential line and the refracted ray at the next successive potential line, by both methods for various angles of incidence and for different intersection angles ε of the tangents drawn from the positions of the incident and refracted ray. By comparing the results obtained by both methods with those derived from the numerical solution of the differential equation of the rays, the author concludes that Snell's construction approximates better the theoretical values than the method of constructing arcs. The numerical result are given in in the text. *N. Chako* (New York, N.Y.).

★**Toraldo di Francia, G. Electromagnetic waves.** Interscience Publishers, New York-London, [1956]. xiii+320 pp. $6.00.

A translation by the author of his Onde elettromagnetiche [Zanichelli, Bologna, 1953; MR 14, 1149].

★**Al'pert, Ya. L. O rasprostranenii èlektromagnitnyh voln nizkoĭ častoty nad zemnoĭ poverhnosteĭ. [On the propagation of electromagnetic waves of low frequency over the earth's surface.]** Izdat. Akad. Nauk SSSR, Moscow, 1955. 112 pp. 4 rubles.

In recent years the need for a satisfactory theory of the phenomenon described by the title has been acutely felt, especially by workers in the fields of communication and meteorology. This little book skillfully adapts a powerful existing theory to the conditions peculiar to low-frequency propagation, with results which go a long way toward meeting that need.

In its general from the problem is to calculate the electromagnetic field of a source in a stratified medium above a spherical earth. If the earth is assumed to be plane, correction for its curvature may be easily made in the final results except near the antipode of the source. The reduced problem of wave propagation in a plane-stratified medium has been treated by L. M. Brehovskih [Izv. Akad. Nauk SSSR. Ser. Fiz. 13 (1949), 505–514, 515–533, 534–545; MR 11, 563, 564], who replaced the given medium by a homogeneous layer bounded by two non-homogeneous half-spaces with reflecting properties, then obtained a solution in the form of a contour integral involving the two "Fresnel reflection coefficients." The coefficient for the earth's surface is taken to be unity. In order to adapt Brehovskih's solution to the problem at hand, the author first obtains expressions for the reflection coefficient of the ionosphere in terms of the ionosphere parameters (height, electron density, collision rate), the frequency, and the complex angle of incidence. Having this expression, he is able to solve graphically the equation whose roots are the poles of the integrand in Brehovskih's integral. The field is then given by the residue sum, of of which only a few terms are needed for satisfactory accuracy for low frequencies.

The results, both subsidiary and final, are thoroughly displayed in many graphs and tables. Of major significance are the calculated field-strength curves for the frequency range 500–30,000 cycles per second and for distances of 100–10,000 kilometers. The theoretical curves show rather good agreement with measurements reported in the literature; in some instances agreement even in detail is achieved. For comparison, the frequently used semi-empirical curves of Espenschied and Austin are plotted on the same graphs within the common domain of applicability. In this comparison the author's results show up to advantage. *R. N. Goss* (San Diego, Calif.).

Nicolau, Edmond. Remarques au sujet des ondes électromagnétiques. Com. Acad. R. P. Române 3 (1953), 365–371. (Romanian. Russian and French summaries)

The author deduces from Maxwell's equations in Cartesian coordinates: If the electric vector depends only on z and t, then (apart from a static field) the magnetic vector does likewise, and both are parallel to the x, y plane. He makes similar observations for cylindrical and spherical waves. *A. Erdélyi* (Pasadena, Calif.).

Grosjean, C. C. Theory of circularly symmetric standing TM waves in terminated irisloaded guides. Nuovo Cimento (10) 2 (1955), 11–26.

The problem of the title is examined by a modified field-matching method similar to the one applied by the author [Nuovo Cimento (10) 1 (1955), 427–438; MR 17, 109] in the case of the finite guide. The validity of certain theorems and some general properties of resonance modes of the finite guide are discussed in detail.

C. J. Bouwkamp (Eindhoven).

Winkler, Gottfried. Zur Theorie der Ausbreitung ebener Wellen in homogenen Plasmen. Ann. Physik (6) 16 (1955), 414–428.

Die Fortpflanzung von ebenen Wellen wird in einem Plasma, einem unbegrenzten Vielteilchensystem elektrisch geladener Teilchen von einheitlichem Standpunkte aus besprochen. Der Verfasser geht dabei weder von der makroskopisch-phenomenologischen, noch von der mikroskopisch-statistischen Methode aus, sondern vereinigt in einer gewissen Weise beide. Den Ausgangspunkt der Berechnungen bilden erstens die beiden Maxwellschen Gleichungen, dann die Kontinuitätsgleichung, sowohl für Elektronen, wie für Ionen und endlich der Lorentzsche

Kraftansatz bzw. die daraus folgenden Bewegungsgleichungen. Die Momentanwerte der in diesen Gleichungen stehenden Grössen, also die der Teilchendichten N_r, der Stromdichte $\mathfrak{J}$, der Verteilungsfunktionen f_r, der Geschwindigkeiten $\mathfrak{V}_r$ und der elektrischen und magnetischen Feldintensität $\mathfrak{E}$ und $\mathfrak{H}$ (wobei sich der Index r entweder auf Elektronen oder auf Ionen bezieht) setzen sich additiv aus den ungestörten Werten dieser Grössen und aus Störungsgliedern zusammen. Mit diesem Ansatz geht der Verfasser in die erwähnten Gleichungen ein und zieht von denen die ungestörten Gleichungen ab. Weiter wird der Ansatz benutzt, dass die Abhängigkeit der Störglieder vom Wellenzahlvektor $\mathfrak{k}$ und der Frequenz ω durch den Faktor $e^{i(\mathfrak{k}\mathfrak{r}+\omega t)}$ beschrieben wird, wo $\mathfrak{k}$ und ω komplex sein können. Das zu lösende mathematische Problem ist, durch Elimination der anderen Veränderlichen einen Zusammenhang zwischen $\mathfrak{k}$ und ω, also eine Dispersionsfunktion herzuleiten. Dieses Problem wird für longitudinale, transversale und gemischte Wellen, letztere sind transversale Wellen mit einer auch longitudinalen Komponente der elektrischen Feldintensität und der Trägergeschwindigkeit, gelöst. Es ergeben sich dabei Feldwellen, Geschwindigkeits- und Dichtewellen. Viele, teilweise in der Literatur schon behandelte Fälle werden aus diesen allgemeinen Resultaten hergeleitet, so z.B. sogar auch die magneto-hydrodynamischen Wellen von Alfvén. Das Problem eines überlagerten Elektronendriftes und das von einem Magnetfelde werden ebenfalls besprochen. *T. Neugebauer* (Budapest).

van Kampen, N. G. On the theory of stationary waves in plasmas. Physica **21** (1955), 949–963.

Bekannterweise versteht man unter einem Plasma ein Gas, das aus Teilchen besteht, die aufeinander mit Kräften von grosser Reichweite (Coulombsche Kräfte) einwirken. Meistens sind diese Teilchen Elektronen, deren Ladungen durch positive Ionen neutralisiert werden, wie das auch in der vorliegenden Arbeit angenommen wird. Ausser den erwähnten sind unter den Teilchen auch Kräfte von kurzer Reichweite wirksam, die Zusammenstösse verursachen. Sind diese Stossprozesse genug häufig, so wird die Theorie der auftretenden Plasmaschwingungen ganz elementar. In der vorliegenden Arbeit befasst sich der Verfasser mit dem entgegengesetzten Fall, d.h. mit dem in dem man diese Zusammenstösse ganz vernachlässigen kann.

Bezeichnet man mit $f_0(\mathbf{v})$ die Verteilungsfunktion der Geschwindigkeiten im Gleichgewichtszustande, mit n_0 die Teilchenzahl, mit e und m die Ladung und die Masse der Teilchen (Elektronen), mit $\mathbf{k}$ den Wellenzahlvektor und mit ω die Frequenz, so folgt nach A. Vlasov [Acad. Sci. USSR. J. Phys. **9** (1945), 25–40; MR **7**, 104]

$$(1) \qquad \frac{4\pi e^2}{m} n_0 \frac{\mathbf{k}}{k^2} \int \frac{\partial f_0}{\partial \mathbf{v}} \frac{d\mathbf{v}}{\varkappa \mathbf{v}-\omega} = 1$$

für den Zusammenhang zwischen $\mathbf{k}$ und ω (Disperionsgleichung). Andere Verfasser erhielten ganz ähnliche Formeln. Leider besitzt jedoch der Integrand in (1) einen Pol; Vlasov nahm deshalb einfach an, dass man in (1) den Cauchyschen Hauptwert zu nehmen hat, ohne jedoch diese Annahme zu begründen. Ziel der vorliegenden Untersuchung ist diese Annahme zu rechtfertigen. Zuerst wird darauf verwiesen, dass die Konvergenzschwierigkeit von (1) nicht eine physikalische sondern eine rein mathematische Frage ist, welche auch tatsächlich gelöst wird. Zu diesem Zwecke wird erstens auch die Diracsche δ-Funktion eingeführt, dann erhält man nach einer Fouriertransformation eine Integralgleichung vom Typ derer, die in letzter Zeit besonders von russischen Mathematikern studiert wurden [vgl. N. I. Muskhelishvili, Singular integral equations, Gostehizdat, Moscow-Leningrad, 1946; MR **8**, 586; **15**, 434]. Als Resultat folgt eine komplette Reihe von stationären Lösungen, die viel zahlreicher als die gewöhnlichen Plasmaschwingungen sind und bei denen k und ω miteinander nicht durch eine Dispersionsgleichung verbunden sind. Spezielle Superpositionen dieser Lösungen liefern dann schwach gedämpfte Wellen, welche der Gleichung (1) genügen. *T. Neugebauer.*

Nicolau, Edmond. Reciprocity and conservation relations in electricity. Rev. Math. Phys. **2** (1954), 9–17 (1955).

A purely formal method is presented for deriving a number of expressions in electromagnetic theory using adjoint operators. Among the expressions thus derived are: the law of conservation of energy of the electromagnetic field, the law of conservation of momentum of the electromagnetic field, and the law of conservation of momentum of transmission lines. A discussion is given of the energy and momentum generated by several sources acting simultaneously. *J. E. Rosenthal.*

Karp, Samuel N.; and Shmoys, Jerry. Calculation of charge density distribution of multilayers from transit time data. Div. Electromag. Res., Inst. Math. Sci., New York Univ., Res. Rep. No. **EM-82** (1955), i+15 pp.

Let $\tau_{12}(\omega)$ denote a known function representing the time required for a radio pulse of carrier frequency ω to travel in the ionosphere between fixed heights x_1 and x_2. Let $\omega_0(x)$ be the unknown critical frequency at height x; $\omega_0^2(x)$ is proportional to the electron density. The integral equation connecting those two functions, $c\tau_{12}(\omega) = \int_{x_1}^{x_2} [1-\omega^{-2}\omega_0^2(x)]^{-\frac{1}{2}} dx$, where c is the velocity of light, has a multiplicity of solutions for $\omega_0(x)$ that is examined by the authors. They show, however, that useful information can be obtained from the equation. Another integral equation, of the type (*) $f(E) = \int_0^\infty \xi(V)(1+V/E)^{-3/2} dV$, is obtained from the first one by introducing changes in variables and by eliminating multiple-valued functions. Here the unknown function $\xi(\omega_0)$ is associated with the sum of the widths of the valleys in the graph of ω_0 against x. Equation (*) is solved with the aid of Mellin transforms. The maximum and minimum values of ω_0, but not their location, together with other properties of the electron density distribution, can be obtained from the function ξ. *R. V. Churchill* (Ann Arbor, Mich.).

Taylor, J. G. Classical electrodynamics as a distribution theory. Proc. Cambridge Philos. Soc. **52** (1956), 119–134.

The author confirms by a more rigorous mathematical discussion the equations obtained previously by Dirac [Proc. Roy. Soc. London. Ser. A. **167** (1938), 148–169] for the classical theory of point charges in interaction with the electromagnetic field, the main feature of which is to involve for the self-force on a particle a finite expression corresponding to the difference between retarded and advanced self-fields. His determination of the potentials and fields makes use of Schwartz' theory of distributions. These quantities are shown to be summable functions even on the world lines of the point charges. The electromagnetic stress tensor, however, is neither a function nor even a distribution in the sense of Schwartz. It is transformed into a distribution over space-time by subtraction of

an infinite part (the 'partie finie' in the sense of Hadamard). The tensor distribution thus obtained furnishes by an argument of momentum-energy conservation Dirac's expression of the self-force. The paper, in addition, brings to light some imperfections of Fremberg's application to the same problem [ibid. **188** (1946), 18–31; MR **8**, 302] of another mathematical algorithm, the analytic continuation method of M. Riesz [Acta Math. **81** (1949) 1–223; MR **10**, 713]. The author does not quote a previous investigation due to S. N. Gupta [Proc. Phys. Soc. Sect. A. **64** (1951), 50–53; MR **12**, 572], and leading to an essentially equivalent redefinition of the stress tensor. Gupta brings out very clearly how this redefinition can be obtained by use of the general subtraction principle of self-effects, also used in the quantum theory.

L. Van Hove (Utrecht).

Piddington, J. H. Electromagnetic field equations for a moving medium with Hall conductivity. Monthly Not. Roy. Astr. Soc. **114** (1954), 638–650 (1955).

In an ionized gas of low density the electrical conductivity is a tensor when a magnetic field is prevalant and the relation between the current (**j**) and the electric (**E**) and magnetic (**H**) fields is given by

(1) $\mathbf{j}=\sigma_0\mathbf{F}_{\parallel}+\sigma_1\mathbf{F}_{\perp}+\sigma_2(\mathbf{H}\times\mathbf{F}_{\perp})/H=\boldsymbol{\sigma}(\mathbf{E}+\mathbf{v}\times\mathbf{H})$,

where $\mathbf{F}_{\parallel}$ and $\mathbf{F}_{\perp}$ are the components of $(\mathbf{E}+\mathbf{v}\times\mathbf{H})$ perpendicular and parallel, respectively, to $\mathbf{H}$, and σ_0, σ_1 and σ_2 are certain constants. In case a uniform magnetic field $\mathbf{H}_0$ (in the z-direction, say) is present and we are interested only in small oscillations, we may combine Maxwell's equations ($4\pi\mathbf{j}=\text{curl}\,\mathbf{H}$; $-\partial\mathbf{H}/\partial t=\text{curl}\,\mathbf{E}$) with equation (1) to give

(2) $$\nabla^2H_z=\frac{\sigma_2}{\sigma_1}\frac{\partial}{\partial z}\,\text{curl}_z\,\mathbf{H}+4\pi\sigma_3\left\{\frac{\partial H_z}{\partial t}-\text{curl}_z(\mathbf{v}\times\mathbf{H})\right\}$$

(3) $$\nabla^2H_x=\frac{\sigma_2}{\sigma_1}\frac{\partial}{\partial z}\,\text{curl}_x\,\mathbf{H}+4\pi\sigma_3\left\{\frac{\partial H_x}{\partial t}-\text{curl}_x\,(\mathbf{v}\times\mathbf{H})\right\}-(1-\sigma_3/\sigma_0)\frac{\partial}{\partial y}\,\text{curl}_z\mathbf{H},$$

and a similar equation for H_y. In equations (2) and (3) $\sigma_3=\sigma_1+\sigma_2^2/\sigma_1$. Reasons are given why under certain practical conditions the term in $(1-\sigma_3/\sigma_0)$ in (2) can be neglected. In the latter case the equations for H_x, H_y and H_z can be written as

(4) $$\nabla^2\mathbf{H}=\frac{\sigma_2}{\sigma_1}\frac{\partial}{\partial z}\,\text{curl}\,\mathbf{H}+4\pi\sigma_3\left\{\frac{\partial\mathbf{H}}{\partial t}-\text{curl}(\mathbf{v}\times\mathbf{H})\right\}.$$

When this equation is combined with the equations of motion appropriate for an incompressible fluid, one finds that transverse hydromagnetic waves [corresponding to solutions of equation (4) of the form $\exp i(pt-kz)$] can be propagated in the direction of the impressed magnetic field H_0. The corresponding "dispersion relation" is

(5) $$\frac{k^2}{p^2}=\left(V^2\pm\frac{\sigma_2 p}{4\pi\sigma_1\sigma_2}+\frac{ip}{4\pi\sigma_3}\right)^{-1},$$

where $V=H_0/(4\pi\varrho)^{\frac{1}{2}}$ denotes the Alfvén velocity. Further, in a compressible medium waves of the form $\exp i(pt-kx)$ (which are analogous to sound waves) can be propagated at right angles to $\mathbf{H}_0$. The corresponding dispersion relation is given by

(6) $$k^4-k^2\left\{\frac{p^2}{U^2}-\frac{ip(U^2+V^2)}{sU^2}\right\}-\frac{ip^3}{sU^2}=0,$$

where U is the velocity of sound and $s=1/4\pi\sigma_3$. In deriving (6), the gas is supposed to change adiabatically during the oscillations. Applications of these results to astrophysical problems are considered.

S. Chandrasekhar (Williams Bay, Wis.).

Lynn, J. W. The tensor equations of electrical machines. Proc. Inst. Elec. Engrs. C. **102** (1955), 149–167.

The paper recapitulates the group-theoretical point of view of rotating electrical machinery used in industry. All machines may be viewed as elements of a finite group and the equations of state (as well as equations of solution) of each machine may be derived from those of any other machine with the aid of a group of non-singular "connection matrices" $C_\alpha{}^\alpha$, by utilizing the laws of transformation of tensors and geometric objects. The latter are associated with a topological "complex" of zero-, one-, two- and three-dimensional spaces (taps, coils, brushes and cylindrical layers of windings) immersed in an n-dimensional non-Riemannian space rigged up with non-holonomic reference frames. Tensors in a topological network (graph) appear as several types of isomorphisms (Maxwell's and Newton's equations) generated by the physics of the problem between the homology sequences of contravariant variables i^α (or x^α) and the cohomology sequences of covariant variables e_α.

In actual practice the equations of any particular machine (having complicated interconnections, sliding contacts and rotating reference axes) are derived, with the aid of a $C_\alpha{}^\alpha$, from those of a hypothetical "generalized" (or "primitive") machine that possesses the simplest possible topological structure. In the latter all component coils and windings are torn apart and are also re-oriented to lie along two sets of axes (stationary or rotating) at right angles in space. The basic equations of this generalized machine in turn are derived by extending the Boltzmann-Hamel form of the dynamical equations of Lagrange, to include a torsion tensor.

G. Kron (Schenectady, N.Y.).

See also: César de Freitas, p. 613; Sternberg, Shipman and Kaufman, p. 671; Samson and Mueller, p. 673; Clauser, p. 676; Graffi, p. 688; Gasanov, p. 692.

Quantum Mechanics

Verde, M. Asymptotic expansions of phase shifts at high energies. Nuovo Cimento (10) **2** (1955), 1001–1014.

The author computes asymptotic formulae for the phase shift of the solution to the Schrödinger equation for scattering in a central, static potential $U(r)$. The expansion is made in inverse powers of a parameter p connected with the energy E of the scattered particle through $p=r_0(2\mu E/\hbar^2)^{\frac{1}{2}}$. Here, r_0 is the range of the potential and μ the mass of the particle. The two main results of the paper are contained in the following formulae.

1) For a potential $U(r)$ that does not become more singular than $U_{-1}/r+U_0+O(r)$ at the origin, the S-shift is given by

$$\tan\delta_0(p)=-\frac{1}{2p}\left[\int_0^\infty\left(U-U_{-1}\frac{\exp[-\alpha r]}{r}\right)dr+U_{-1}\log\frac{2p}{\alpha}\right]$$
$$-\left(\frac{1}{2p}\right)^3\left[\left(U-U_{-1}\frac{\exp[-\alpha r]}{r}\right)_{r=0}+2U_{-1}U_0\log\frac{2p}{\alpha}\right.$$
$$\left.+2\int_0^\infty\left(U(rU)'-\frac{U_{-1}U_0}{r}\exp[-\alpha r]\right)dr\right]+O\left(\frac{1}{ps}\right).$$

This formula can also be used for regular potentials when the coefficient U_{-1} vanishes. The quantity α is an arbitrary parameter.

2) For a potential that is regular at the origin, the phase shift for an arbitrary angular momentum l is given as

$$\tan \delta_l(p) = -\frac{1}{2p}\Big[\int_0^\infty U(r)-f(r))dr+\frac{1}{\alpha}\Big((2U(0)+\frac{U'(0)}{\alpha}\Big)\Big]$$
$$-\Big(\frac{1}{2p}\Big)^3\Big[\int_0^\infty U^2dr+2l(l+1)\Big(\int_0^\infty \frac{1}{r^2}(U-f)dr-\alpha U(0))$$
$$+2l(l+1)U'(0)\log\frac{2p}{\alpha}-\tfrac{1}{2}+\sum_1^l \frac{1}{n}\Big)\Big]+O\Big(\frac{1}{ps}\Big),$$
$$f(r)=[U(0)+r(U'(0)+\alpha U(0)]\exp[-\alpha r].$$

It might perhaps be remarked that if one puts $l=0$ in this last formula, one does not obtain the same result as if U_{-1} is put equal to zero in the first one. *G. Källén.*

Costa de Beauregard, Olivier. Diffraction par une ouverture plane à contour variable. Interprétation physique des précédentes formules. C. R. Acad. Sci. Paris **242** (1956), 467–470.

Gasanov, R. G. On a boundary problem of quantum mechanics. Akad. Nauk Azerbaĭdžan. SSR. Trudy Inst. Fiz. Mat. 3 (1948), 44–52. (Russian. Azerbaijanian summary)

The problem is that of electron scattering by a plane metal surface x_1Ox_2 considered as a crystal. Inside the crystal lattice the potential U is assumed to have the form $U_0-\sum_{i=1}^3 U_i \cos(2\pi x_i/a_i)$, where a_i is the length of the side of the unit lattice cell in the direction of the coordinate x_i. Outside the crystal ($x_3>0$), U is zero. The wave function ψ satisfies the equations $\Delta\psi+\delta E_n\psi=0$ outside and $\Delta\psi_i+\delta(E_i-U_i)\psi_i=0$ inside the crystal, where δ is a constant, E_n is the energy of the unperturbed electron, and E_i is the energy of the electron perturbed by interaction with the primary field. At the boundary ψ and $\partial\psi/\partial x_3$ are to be continuous. The problem reduces to the solution of a pair of Fredholm integral equations of first kind. On expanding the known functions in powers of x_1 and x_2, one obtains infinite series representations for the wave amplitudes. This same problem has been considered by P. M. Morse [Phys. Rev. (2) **35** (1930), 1310–1324]. *R. N. Goss* (San Diego, Calif.).

Lipkin, H. J.; de Shalit, A.; and Talmi, I. On the description of collective motion by the use of superfluous coordinates. Nuovo Cimento (10) 2 (1955), 773–798.

This paper gives a very clear and careful discussion of the problems raised by the consideration of collective modes of motion in the nuclear shell model [A. Bohr and B. R. Mottelson, Danske Vid. Selsk. Mat.-Fys. Medd. **27** (1953), no. 16]. The authors choose the approach by which the collective motion is described by extra coordinates (in addition to the particle coordinates), so that the Hamiltonian and the wave function contain superfluous variables which however should not affect physical quantities. The method is first presented in detail for the simplest case of collective motion, i.e. the motion of the center-of-mass. Beyond preparing the more complicated cases, this discussion describes in which sense the center-of-mass motion is eliminated in the usual shell model, an elimination often tacitly assumed in the conventional presentations. The case of collective rotation is then discussed at length. The paper ends with a short sketch of the method of superfluous coordinates in its general form. As mentioned by the authors in a footnote, this method has been studied independently by Villars. *L. Van Hove* (Utrecht).

★**Klein, O. Quantum theory and relativity.** Niels Bohr and the development of physics, pp. 96–117. McGraw-Hill Book Co., New York, N. Y., 1955. $4.50.

A program for a quantum field theory is proposed, in which the operators used should be connected with a general transformation group containing the space-time coordinate transformations of the general relativity theory as a subgroup, the group being characterized by the quantum conditions. As an example, but in the framework of the special relativity theory, the quantization of the electromagnetic field is carried out: the commutation relations are chosen so that field variables at different points of space-time are independent, and the field equations are regarded as conditions on the state vectors. A corresponding treatment of the Dirac electron field is also given. *N. Rosen* (Haifa).

★**Rosenfeld, L. On quantum electrodynamics.** Niels Bohr and the development of physics, pp. 70–95. McGraw-Hill Book Co., New York, N. Y., 1955. $4.50.

The measurability and the fluctuations of electromagnetic fields and of charge and current densities are discussed, and their relation to the formalism of quantum electrodynamics is considered. Recent theoretical developments are surveyed, and some of the difficulties existing in the electromagnetic and meson field theories are pointed out. *N. Rosen* (Haifa).

★**Landau, L. D. On the quantum theory of fields.** Niels Bohr and the development of physics, pp. 52–69. McGraw-Hill Book Co., New York, N. Y., 1955. $4.50.

Some questions concerning convergence and renormalization in quantum electrodynamics are considered. Among other things, it is found, with certain assumptions, that as the cut-off parameter tends to infinity (point electron), the observable electron charge tends to zero no matter how large one takes the "intrinsic" electron charge. The situation in meson field theory is also discussed, and the essential difference between the results in the cases of pseudoscalar and pseudovector coupling is emphasized. *N. Rosen* (Haifa).

★**Pauli, W. Exclusion principle, Lorentz group and reflection of space-time and charge.** Niels Bohr and the development of physics, pp. 30–51. McGraw-Hill Bock Co., New York, N. Y., 1955. $4.50.

This paper discusses "weak reflection" in which all four space-time coordinates undergo a change in sign, particle-antiparticle conjugation, and "strong reflection" in which both of the preceding changes are carried out. The behavior of various quantized fields under these transformations are considered, as well as the restrictions on the Lagrangian density imposed by the requirement of invariance. A general proof is given that, in the case of strong reflection, invariance follows from proper Lorentz invariance and the transformation of a quantity is uniquely determined by its spinor or tensor character. *N. Rosen* (Haifa).

Taniuti, Tosiya. On the theories of higher derivative and non-local couplings. I. Progr. Theoret. Phys. 13 (1955), 505–521.

The initial-value problem is examined for a classical harmonic oscillator, with non-local interaction, and with interactions involving higher derivatives. It is shown that for both these cases a solution in powers of the coupling constant may be developed, based on the position and the velocity at the initial time. It is also considered how the characteristic cone is shifted from the light cone for relativistic field theories with non-local interactions or interactions involving higher derivatives.

P. T. Matthews (Birmingham).

Visconti, A. Applications à théorie quantique des champs d'une solution exacte de certaines équations linéaires opératorielles. J. Phys. Radium (8) 16 (1955), 1–15.

The author develops the theory of linear integral equations in Hilbert space with a view to applications in Field Theory, particularly to spinor-particle scattering in an external field. The results of Feynman [Phys. Rev. (2) 76 (1949), 749–759] and Salam and Matthews [ibid. 90 (1953), 690–695; MR 15, 82] are rederived. A generalisation of the Fredholm method leads to a discussion of the Nörlund sum of a divergent series and its relation to the conjectured asymptotic behaviour of the S-matrix.

A. Salam (Cambridge, England).

Matthews, P. T.; and Salam, A. Propagators of quantized field. Nuovo Cimento (10) 2 (1955), 120–134.

This paper gives a systematic account of the method of functional integration in connection with the theory of the "Green's functions" or "propagators" in quantized field theories. The connection between the functional integration and the "sum over all paths" of Feynman [Rev. Mod. Phys. 20 (1948), 367–387; MR 10, 224] is discussed and explicit calculations are carried through for some simple examples with only one quantized field. The somewhat elusive concept of a functional integral is given a meaning in the following way. The functions $\varphi(x)$ appearing in a functional integral of the form

$$\Delta(1,2)=\frac{1}{M}\int\varphi(1)\varphi(2)\exp[-iI_m]\delta\varphi, \tag{1}$$

$$M=\int\exp[-iI_m]\delta\varphi, \tag{1a}$$

$$I_m=-\tfrac{1}{2}\int\varphi(x)\mathfrak{K}(x,y)\varphi(y)dxdy \tag{1b}$$

are expanded in a complete set of functions $\varphi_n(x)$ satisfying

$$\tfrac{1}{2}\int\varphi_n(x)\mathfrak{K}(x,y)\varphi_m(y)dxdy=\delta_{nm}. \tag{2}$$

As the denominator M in (1) is singular if the integral (2) does not exist, it is argued that only functions that can be expanded in the form $\varphi(x)=\sum_n a_n\varphi_n(x)$ give a non-vanishing contribution to (1). No discussion of the fact that the same exponential appears also in the numerator is given. The expansion coefficients a_n are then treated as independent variables and (1) is written

$$\Delta(1,2)= \tag{3}$$
$$[\prod_r N(a_r)^{-1}\sum_{ij}\varphi_i(1)\varphi_j(2)\int a_ia_j\exp[-i\sum_s a_s^2]\prod_t da_t,$$

$$N(a)=\int\exp[-ia^2]da. \tag{3a}$$

For the simple examples treated explicitly it is shown that this method gives results in agreement with what has been obtained earlier in other ways. General formulae are then written down also for the case of two interacting quantized fields and it is stated that: "If these integrals can be evaluated they determine directly the scattering amplitudes".

G. Källén (Copenhagen).

Källén, G.; and Sabry, A. Fourth order vacuum polarization. Danske Vid. Selsk. Mat.-Fys. Medd. 29, no. 17 (1955), 20 pp.

This is an attractive calculation of the fourth-order vacuum polarization $\Pi(p^2)$. It is remarked that $p^2\Pi(p^2)$ is proportional to

$$\sum_s (0|j_\mu^1|z)(z|j_\mu^1|0)+\{(0|j_\mu^2|z)(z|j_\mu^0|0)+\text{complex conjugate}\},$$

where j_μ^0, j_μ^1 etc. are the successive terms in the expansion

$$j_\mu=ej_\mu^0+e^2j_\mu^1+e^3j_\mu^2+\cdots.$$

j_μ^1 and j_μ^2 have already been computed by Källén [Ark. Fys. 2 (1950), 371–410; MR 12, 890] and Schwinger [Phys. Rev. (2) 76 (1949), 790–817; MR 11, 569]. The calculation of the 4th-order vacuum polarization thus reduces to the evaluation of the sum $\sum_s$. In this physically perspicuous computation, all difficulties connected with "regularization" and "over-lapping divergences" are completely circumvented. The final results agree with those of Baranger, Dyson and Salpeter [ibid. 88 (1952), 680].

A. Salam (Cambridge, England).

Ebel, Marvin E. Causal behaviour of field theories with non-localizable interactions. Danske Vid. Selsk. Mat.-Fys. Medd. 29 (1954), no. 2, 31 pp.

Fierz [Helv. Phys. Acta 23 (1950), 731–739; MR 12, 573] has shown that the positive-frequency part of the Feynman function $\Delta_F(k)$ propagates only into the forward light cone, except for a part which damps out rapidly, and is unobservable due to the complementarity existing between time and energy. Physically the result underlines the causal nature of the local field theories; mathematically it arises from the fact that the poles of Δ_F lie in the second and the fourth quadrants in the complex k_0-plane.

In the present paper the author considers, in this context, a non-local interaction of the type introduced by Kristensen and Møller [Danske Vid. Selsk. Mat.-Fys. Medd. 27 (1952), no. 7], with a form factor in the Lagrangian. The causality conditions is formulated thus: to the extent to which the energy of a particle is known to be positive and to the extent to which its points of creation and destruction may be determined, these points must be separated by a time-like distance, and the point of destruction must occur later than the point of creation. This leads to the following conclusions: (a) The poles of the propagation function Δ_F' must lie only in the second and the fourth quadrants. (b) The form function $G(k_1, k_2)$ should be a sufficiently smooth function of the variables k_1^2, k_2^2 and $(k_1+k_2)^2$. More precisely, the probability of observing signals transmitted with velocities greater than light decreases more rapidly than the inverse $(n+4)$th power of the spatial distance between the points of observation, if the function G has continuous derivatives of the nth order.

A. Salam.

Rzewuski, J. Relativistic quantum dynamics of a system of interacting particles. Acta Phys. Polon. 13 (1954), 29–43. (Russian summary)

Relativistic systems of particles interacting by means of

retarded (or advanced) forces are considered. The interaction is assumed small, the co-ordinate and momentum operators of the particles being expanded in powers of the interaction constant. In the lowest oder of approximation, these obey the equations of motion for non-interacting particles and can be quantized directly. Higher-order approximations are expressed in terms of the lowest order, and the quantization scheme carried through, for this scheme of relativistic quantum dynamics.

A. Salam (Cambridge, England).

Sunakawa, Sigenobu. The formal theory of scattering. Progr. Theoret. Phys. **14** (1955), 175–197.

In time-dependent scattering theory it is necessary to "switch on" the interaction adiabatically, or to average over the distant past as suggested by Gell-Mann and Goldberger [Phys. Rev. (2) **91** (1953), 398–408; MR **15**, 382]. The author introduces a third method which is also a switching-on of the interaction, but whose definition differs from the usual one. With this method the general definition of the S-matrix which also permits bound states is repeated. All three methods are rather formal and not very satisfactory from a physical point of view. On the other hand, it has been known for some time [see e.g., M. N. Hack, ibid. **96** (1954), 196–198] that the description of incident particles by wave packets rather than by plane waves might give a better physical description; mathematically it leads to essentially the same limiting procedure for defining certain functions for times $t\to\pm\infty$. The author shows explicitly that this is indeed the case; he works with suitable wave packets for non-relativistic particles. If v is the velocity of a particle and L the width of the corresponding wave packet, then v/L takes the place of the usual switching-on parameter ε. Rearrangement scattering is treated as an example. The paper also contains a criticism of the Gell-Mann-Goldberger method which seems to indicate that the author is not acquainted with the equivalence proof for the first two methods mentioned above [F. Coester, M. Hamermesh, and K. Tanaka, ibid. **96** (1954), 1142–1143; MR **16**, 432].

F. Rohrlich (Iowa City, Ia.).

Reifman; Alfred, DeWitt, Bryce S.; and Newton, Roger G. Relations between bound-state problems and scattering theory. Phys. Rev. (2) **101** (1956), 877–879.

The nuclear many-body bound state problem has been thought of as follows. For any two particles i and j in a large nucleus interacting via a short-range potential $v(ij)$ there exists an effective potential $V_C(ij)=V_C(i)+V_C(j)$ due to all the other particles in the nucleus. This model leads to a system of equations given by Brueckner [Phys. Rev. (2) **97** (1955), 1353–1366] which in the present paper are compared to the corresponding scattering problem. Let $H_0(ij)=H_0(i)+H_0(j)$ be the kinetic energy of the particles, so that $[H_0(ij)+V_C(ij)]\psi_{ij}^{(0)}=E_{ij}^{(0)}\psi_{ij}^{(0)}$ and $[H_0(ij)+V_C(ij)+v(ij)]\psi_{ij}=E_{ij}\psi_{ij}$. The energy shift $\Delta E_{ij}=E_{ij}-E_{ij}^{(0)}$ is then the diagonal matrix element $(\psi_{ij}^{(0)}, R_B^{ij}(E_{ij})\psi_{ij}^{(0)})$ of an operator R_B^{ij} defined by $v(ij)\psi_{ij}=R_B^{ij}(E_{ij})\psi_{ij}^{(0)}$. The limit is now discussed, as the nuclear volume becomes arbitrarily large, leaving the depth of $V_C(ij)$ constant. It is argued that in this limit $\Delta E_{ij}\to 0$ and $R_B^{ij}\to K^{ij}$, the usual K-matrix, which satisfies $K^{ij}=v(ij)+v(ij)P[E_{ij}-H_0(ij)-V_C(ij)]^{-1}K^{ij}$, where P denotes the Cauchy principal value. This is Brueckner's first equation. The relation to the corresponding scattering problem which involves the equation

$$R_+^{ij}=v(ij)+v(ij)\lim_{\varepsilon\to+0}[E_{ij}-H_0(ij)-V_C(ij)+i\varepsilon]^{-1}R_+^{ij}$$

corresponding to different boundary conditions is apparent. The diagonal elements of R_+ give the forward scattering amplitudes. It is noted that K is not the real part of R_+, except in second-order perturbation theory. The second of Brueckner's equations can now be written $V_C(i)=\sum_{j\neq i}\Delta E_{ij}$ which remains finite and non-zero in the above mentioned limit, because the nuclear density is held constant. Finally, it is remarked that the limit of R_B used by Deser, Goldberger, Baumann, and Thirring [ibid. **96** (1954), 774–776] and which is not K, is only correct in the zero-energy limit. *F. Rohrlich.*

Nishijima, Kazuhiko. Solutions of a Bethe-Salpeter equation for scattering states. Progr. Theoret. Phys. **14** (1955), 203–213.

The subject of the paper is the Bethe-Salpeter equation for two scalar particles interacting through a massless scalar field. A complete set of bound-state solutions for this equation was obtained recently by Wick [Phys. Rev. (2) **96** (1954), 1124–1134; MR **16**, 655] and Cutkosky [ibid. **96** (1954), 1135–1141; MR **16**, 656]. The present paper uses the same method to obtain scattering solutions. As the bound-state solutions showed a behavior in the degeneracy of their eigenvalues which was similar to the non-relativistic hydrogen atom, the scattering solutions exhibit similarities to the non-relativistic Rutherford scattering. The method used is that of integral transformation by parametric representation, and the relationship to Cutkosky's results is shown at every step. An additional feature of the present article is a discussion of the virtual levels which are given by the eigenstates of the homogeneous parametric integral-equation form of the Bethe-Salpeter equation for scattering states. It is also shown explicitly that resonance scattering takes place near such a virtual level. Thus with the results of this paper we have a complete set of solutions for at least one case of the Bethe-Salpeter equation. *M. J. Moravcsik.*

See also: Matsubara, p. 695.

Thermodynamics, Statistical Mechanics

Rosenfeld, L. On the foundations of statistical thermodynamics. Acta Phys. Polon. **14** (1955), 3–39. (Russian summary)

The author gives an excellent analysis of our present state of knowledge and understanding concerning the foundations of statistical mechanics and the statistical interpretation of thermodynamics. The emphasis lies on the sort of complementarity existing between the aims of a thermodynamical description and a microscopic description of systems with many degrees of freedom. The analogy as well as the differences with the quantum-mechanical measuring process are stressed. The technical discussion starts with isolated systems and describes the classical and quantummechanical ergodic theorems. The statistics of non-isolated systems is then derived from the case of isolated ones. The statistical foundation of thermodynamics is finally discussed.

L. Van Hove (Utrecht).

Fuchs, W. H. J. On the virial series of the ideal Bose-Einstein gas. J. Rational Mech. Anal. **4** (1955), 647–652.

The analytical properties of the virial series (pressure expressed in power series of the density) of an ideal Bose-Einstein gas are investigated. The radius of convergence is shown to be finite, lower and upper bounds are given for its value. The series is further shown to define an analytic function which is exempt of singularities in some strip containing the whole real axis. The critical density of Bose-Einstein condensation lies inside the convergence circle of the virial series. It is unrelated to singularities of the analytic function defined by the series.
L. Van Hove (Utrecht).

Shapiro, H. S. Virial series of the ideal Bose-Einstein gas. Phys. Rev. (2) **99** (1955), 1673–1674.

The virial series of the ideal Bose-Einstein gas is shown to have a finite radius of convergence R, in contradiction with the conjecture of Widom [Phys. Rev. (2) **96** (1954), 16–17; MR **16**, 205] that R should be infinite. This result is in agreement with, but much weaker than, the result of the paper reviewed above, giving upper and lower bounds for R and implying that condensation occurs for a density interior to the convergence circle. This result of Fuchs is quoted in the present paper as a footnote added after submission. *L. Van Hove* (Utrecht).

Koga, Toyoki. Extended interpretation of the Boltzmann-Maxwell equation (A theory of nonstationary random processes). J. Chem. Phys. 23 (1955), 2275–2281.

The first part of this paper is a commentary on the limitations of the theories of nonstationary processes, especially the works of Born, H. S. Green, Kirkwood and M. S. Green. Most of the arguments are qualitative and do not appear to contain any new ideas. The author proposes to extend these theories by "coarse-graining" in the phase space of the complete system. This is claimed to be a new method although this reviewer fails to see any real difference between it and the scheme described for example by Tolman [The principles of statistical mechanics, Oxford, 1938]. The method is applied to rarefied gases to give the Boltzmann-Maxwell (B-M) equation, but in the process of deriving the equation, the author assumes it is valid without the coarse-graining. He concludes from this that the B-M equation is valid under a wider interpretation in which one averages over a series of observations. This interpretation, however, is basic to all results of statistical mechanics. Although the B-M equation was derived, interpreted and discussed in great detail long before the modern theories of Born, Green, etc., no reference is made to these earlier works. *G. Newell.*

Matsubara, Takeo. A new approach to quantum-statistical mechanics. Progr. Theoret. Phys. **14** (1955), 351–378.

The density matrix ϱ of a canonical ensemble satisfies the equation

$$(*) \qquad -\partial\varrho/\partial\beta = H\varrho = (H_0+H_1)\varrho,$$

where H is the Hamiltonian of the total system, and $\beta=1/kT$. Because of the similarity of (*) with the Schrödinger equation, it is possible to adapt the methods of quantum field theory to the calculation of ϱ from (*). The perturbation treatment is developed for the case of an electronphonon system, H_1 being the electron-phonon interaction. This leads to ordered products, modified S and D functions, Feynman graphs, and renormalization of the phonon mass. Some of the lowest-order terms are given explicitly and found to agree with known results. The non-perturbation treatment of Peierls and Schwinger, based on Green's functions and their integral equations, is also shown to be applicable. Finally the method is applied to a system of many particles interacting through a two-body potential. *N. G. van Kampen* (Utrecht).

Potts, R. B.; and Ward, J. C. The combinatorial method and the two-dimensional Ising model. Progr. Theoret. Phys. **13** (1955), 38–46.

The present paper completes certain parts of the determinantal discussion of Kac and Ward [Phys. Rev. (2) **88** (1952), 1332–1337] and extends the method to consider the correlations between occupancy of a given site and related sites. The discussion is based in detail on the earlier paper. The additional correlation functions are obtained by bordering the previous determinants.
F. J. Murray (New York, N.Y.).

Mazur, P.; Montroll, E. W.; and Potts, R. B. Effect of defects on lattice vibrations. II. Localized vibration modes in a linear diatomic chain. J. Washington Acad. Sci. **46** (1956), 2–11.

Methods developed recently by Montroll and Potts [Phys. Rev. (2) **100** (1955), 525–593; MR **17**, 568] are used in a study of the normal modes and frequencies of a long diatomic linear chain with a particle of abnormal mass at one of the lattice positions, or with two such defects. The normal lattice is taken to consist of masses m, M, in alternation, with identical springs connecting nearest neighbors. Substitution of m' for one of the m particles gives rise, in three of the four possible cases, to localized modes with discrete frequencies in formerly forbidden ranges: (1) If $m'<m<M$, one mode is displaced from the upper or optical band into the region above. (2) If $m'>m$, $m<M$, one mode goes from the optical band into the gap between optical and acoustical bands. (3) If $m'<m$, $m>M$, one mode goes from the optical band into the region above and another goes from the acoustical band into the gap. (4) If $m'>m$, $m>M$, no discrete modes emerge.

The change of zero-point energy produced by the substitution of an m' for an m is calculated for the cases $m\gg M$ and $m\approx M$. When two abnormal masses are substituted at a distance l from each other, they have an interaction energy which is $\sim l^{-3}$ as $l\to\infty$ and which is attractive if both substituents are heavier or both lighter than the masses they replace, and repulsive if one is lighter, the other heavier. *C. Herring.*

See alo: Putnam, p. 637; Beer, Chase and Choquard, p. 672.

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